A point-to-set principle for separable metric spaces

Elvira Mayordomo

Universidad de Zaragoza, Iowa State University

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Why do we effectivize a mathematical concept $A$?

- Because $\text{eff}A$ can be useful when dealing only with effective objects
- Because $\text{eff}A$ approximates $A$ (hopefully in a known and useful way)
- Because for interesting and simple objects $\text{eff}A$ is equivalent to the classical concept $A$
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Is this all?
Why do we effectivize a mathematical concept $A$?

Because relativization of $\text{eff}A$ gives you back $A$.
Today

- Hausdorff dimension
- Lutz’s effective dimension for Cantor and Euclidean spaces
- Point-to-set principle
- Effective dimension and point-to-set principle in separable spaces
Hausdorff definition of dimension

Let $\rho$ be a metric on a set $X$.

- For $E \subseteq X$ and $\delta > 0$, a $\delta$-cover of $E$ is a collection $\mathcal{U}$ such that for all $U \in \mathcal{U}$, $\text{diam}(U) < \delta$ and
  \[ E \subseteq \bigcup_{U \in \mathcal{U}} U. \]

- For $s \geq 0$,
  \[ H^s(E) = \lim_{\delta \to 0} \inf_{\mathcal{U} \text{ is a } \delta\text{-cover of } E} \sum_{U \in \mathcal{U}} \text{diam}(U)^s \]

The Hausdorff dimension of $E \subseteq X$ is
\[ \dim_H(E) = \inf \{ s \mid H^s(E) = 0 \}. \]
Definition
For a finite string $w$, and a universal Turing machine $U$, 

$$K_U(w) = \{|p| \mid U(p) = w\}$$

This concept is invariant on $U$ up to an additive constant, we drop the $U$. 
Lutz’s effective Hausdorff dimension: Kolmogorov complexity

Definition
For a finite string $w$, and a universal Turing machine $U$, $K_U(w) = \{ |p| | U(p) = w \}$

This concept is invariant on $U$ up to an additive constant, we drop the $U$. $K(w)$ is the length of the shortest description from which $w$ can be computably recovered.
Effective dimension in Cantor space

- \( \{0, 1\}^\infty \) is the set of infinite binary sequences
- For \( x \in \{0, 1\}^\infty \), \( x \upharpoonright n \) the length \( n \) finite prefix of \( x \)

**Definition**

For every \( x \in \{0, 1\}^\infty \), \( E \subseteq \{0, 1\}^\infty \),

\[
\text{cdim}(x) = \liminf_n \frac{K(x \upharpoonright n)}{n}.
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- Lutz original definition of constructive dimension uses gambling, this is a characterization.
Effective dimension in Cantor and Euclidean spaces

- Very robust concepts, they can be defined using
  - measure theory
  - gambling
  - information theory

- Resource-bounded versions are natural and useful

- It is non-necessarily zero and meaningful on singletons.

- **By absolute stability effective dimension can be investigated in terms of the dimension of individual points.**

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- **By absolute stability effective dimension can be investigated in terms of the dimension of individual points.**
- For $\Sigma^0_2$ sets, constructive dimension is exactly Hausdorff dimension ... For a while this was good enough
Classical dimension can be characterized in terms of effective dimension (point-to-set principle)
How effective dimension can be used for classical geometry

Classical dimension can be characterized in terms of effective dimension (point-to-set principle)

Theorem (J.Lutz, N.Lutz 2017)

For every $E \subseteq \{0, 1\}^\infty$,

$$\dim_H(E) = \min_{B \subseteq \{0, 1\}^*} \cdim^B(E).$$

This theorem allows us to prove classical dimension results using Kolmogorov complexity, already a few very interesting ones (N.Lutz-Stull on generalized Furstenberg sets, N. Lutz on the intersection formula)

We can now investigate the dimension of a set in terms of the dimension of its points
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(Marstrand 1954, Mattila 1975) Let $E \subseteq \mathbb{R}^n$ be an analytic set. Then for almost every direction $e$

$$\dim_H(\text{proj}_e E) = \min\{\dim_H(E), 1\}.$$ 

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(N. Lutz 2017) Let $E \subseteq \mathbb{R}^n$ be a set with $\dim_H(E) = \dim_p(E)$. Then for almost every direction $e$

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Example of a result using the point-to-set principle

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We want to use the point to set principle in other spaces
Let \((X, \rho)\) be a separable metric space. Let \(D\) be a countable dense set and \(f : \{0, 1\}^* \rightarrow D\) be surjective.
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Kolmogorov complexity in a separable space

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**Definition**

Let $x \in X$, $r \in \mathbb{N}$. The *Kolmogorov complexity of $x$ at precision $r$* is

$$K^f_r(x) = \inf \{ K(w) \mid \rho(x, f(w)) \leq 2^{-r} \}.$$
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This is an extension of the notion used for reals. For computable metric spaces the idea is inherent in (Melnikov Nies 2013) work on K-trivials.
Effective dimension in a separable space

$(X, \rho)$ is a separable metric space, $D$ is a countable dense set, and $f : \{0, 1\}^* \to D$ surjective

**Definition**

Let $x \in X$,

$$\text{cdim}^f(x) = \liminf_r \frac{K_r^f(x)}{r}. $$

Let $E \subseteq X$,

$$\text{cdim}^f(E) = \sup_{x \in E} \text{cdim}^f(x).$$
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Both definitions relativize to any oracle \(B\) by using \(K^B(w)\).
Let $X$ be a separable metric space. Let $D$ be a countable dense set and $f : \{0, 1\}^* \to D$ be surjective.

**Theorem (Main result)**

Let $E \subseteq X$. Then

$$\dim_H(E) = \min_{B \subseteq \{0,1\}^*} \cdim^{f,B}(E).$$
Example: Hilbert cube

Let $\mathcal{H} = [0, 1]^\mathbb{N}$ with metric

$$\rho((a_n), (b_n)) = \sum_n |b_n - a_n|2^{-n}.$$
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$$S = \{(a_n) | a_n = 0 \text{ a.e. } n\}.$$
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$$K_r((a_n)) \leq \sum_{i=1}^{r+1} 2K_{r+1}(a_i).$$
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- It is clear that for some $(a_n)$, $K_r((a_n)) \geq r^2$, so there are points of infinite $\text{cdim}$ in $\mathcal{H}$.
- Examples of points with finite dimension, when $(a_n)$ is bounded.
- For unbounded $f$, there is $a_n \leq f(n)$ with $\text{cdim}((a_n)) = \infty$. 
Conclusions

- Lutz effectivization of Hausdorff dimension can be generalized to all separable metric spaces via Kolmogorov complexity.
- The point-to-set principle allows us to capture classical Hausdorff dimension through the pointwise analysis of the dimension of sets.
- Let us use it to solve open problems in fractal geometry.
References

- Elvira Mayordomo, A point-to-set principle for separable metric spaces, in preparation
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- It can be done from measure
- For spaces with a suitable regularity condition it can be done through gambling (pretty useful for resource-bounds)
- Similarly **packing dimension** and **exact dimension** can be effectivized for all separable spaces
Let $g$ be increasing in both arguments. For $s \geq 0$,
\[ H_{g,s}(E) = \lim_{\delta \to 0} \inf_{\mathcal{U}} \text{is a } \delta\text{-cover of } E \sum_{U \in \mathcal{U}} g(s, \text{diam}(U)) \]

\[ \dim_{H}^{(g)}(E) = \inf \{ s \mid H_{g,s}(E) = 0 \} . \]

**Definition**

Let $X$ be a separable metric space. Let $x \in X$,
\[ \text{cdim}_{g}^{f}(x) = \inf \left\{ s \mid \exists \infty r \ K_{r}^{f}(x) \leq -\log(g(s, 2^{-r})) \right\} . \]

**Theorem**

Let $E \subseteq X$. Then
\[ \dim_{H}^{(g)}(E) = \min_{B \subseteq \{0,1\}^*} \text{cdim}_{g,B}^{f}(E). \]