A point-to-set principle for separable metric spaces

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Why do we effectivize?

Why do we effectivize a mathematical concept A?

- Because *effA* can be useful when dealing only with effective objects
- Because *effA* approximates A (hopefully in a known an useful way)
- Because for interesting and simple objects *effA* is equivalent to the classical concept A

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Is this all?

Why do we effectivize a mathematical concept A?

Because relativization of $effA$ **gives you back** A

Today

- Hausdorff dimension
- Lutz's effective dimension for Cantor and Euclidean spaces
- Point-to-set principle
- Effective dimension and point-to-set principle in separable spaces

Hausdorff definition of dimension

Let ρ be a metric on a set X.

- For $E \subseteq X$ and $\delta > 0$, a δ-cover of E is a collection $\mathcal{U}$ such that for all $U \in \mathcal{U}$, $\mathrm{diam}(U) < \delta$ and
$$E \subseteq \bigcup_{U \in \mathcal{U}} U.$$
- For $s \geq 0$,
$$H^s(E) = \lim_{\delta \to 0} \inf_{\mathcal{U} \text{ is a } \delta\text{-cover of } E} \sum_{U \in \mathcal{U}} \mathrm{diam}(U)^s$$

The Hausdorff dimension of $E \subseteq X$ is
$$\dim_{\mathrm{H}}(E) = \inf \{ s \mid H^s(E) = 0 \}.$$

Lutz's effective Hausdorff dimension: Kolmogorov complexity

Definition

For a finite string w, and a universal Turing machine U,

$$\mathrm{K}_U(w) = \{|p| \mid U(p) = w\}$$

This concept is invariant on U up to an additive constant, we drop the U.

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$\mathrm{K}(w)$ **is the length of the shortest description from which** w **can be computably recoverered.**

Effective dimension in Cantor space

- $\{0,1\}^\infty$ is the set of infinite binary sequences
- For $x \in \{0,1\}^\infty$, $x \upharpoonright n$ the the length n finite prefix of x

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- Lutz original definition of constructive dimension uses gambling, this is a characterization.

Effective dimension in Cantor and Euclidean spaces

- Very robust concepts, they can be defined using
  - measure theory
  - gambling
  - information theory
- Resource-bounded versions are natural and useful
- It is non necessarily zero and meaningful on singletons.
- **By absolute stability effective dimension can be investigated in terms of the dimension of individual points.**
- For Σ_2^0 sets, constructive dimension is exactly Hausdorff dimension ...

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Theorem (J.Lutz, N.Lutz 2017)

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- **This theorem allows us to prove classical dimension results using Kolmogorov complexity**, already a few very interesting ones (N.Lutz-Stull on generalized Furstenberg sets, N. Lutz on the intersection formula)
- We can now investigate the dimension of a set in terms of the dimension of its points

Example of a result using the point-to-set principle

Theorem

(Marstrand 1954, Mattila 1975) Let $E \subseteq \mathbb{R}^n$ be an **analytic set**. *Then for almost every direction e*

$$\dim_{\mathrm{H}}(\mathrm{proj}_e E) = \min\{\dim_{\mathrm{H}}(E), 1\}.$$

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We want to use the point to set principle in other spaces

Kolmogorov complexity in a separable space

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Definition

Let $x \in X, r \in \mathbb{N}$. The Kolmogorov complexity of x at precision r is

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This is an extension of the notion used for reals. For computable metric spaces the idea is inherent in (Melnikov Nies 2013) work on K-trivials

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Both definitions relativize to any oracle B by using $\mathrm{K}^B(w)$

Point to set principle for separable X

Let X be a separable metric space. Let D be a countable dense set and $f : \{0,1\}^* \to D$ be surjective.

Theorem (Main result)

Let $E \subseteq X$. Then

$$\dim_{\mathrm{H}}(E) = \min_{B \subseteq \{0,1\}^*} \mathrm{cdim}^{f,B}(E).$$

Example: Hilbert cube

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$$S = \{(a_n) \,|\, a_n = 0 \text{ a.e.} n\}.$$

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- Examples of points with finite dimension, when (a_n) is bounded.
- For unbounded f, there is $a_n \leq f(n)$ with $\mathrm{cdim}((a_n)) = \infty$.

Conclusions

- Lutz effectivization of Hausdorff dimension can be generalized to all separable metric spaces via Kolmogorov complexity
- The point-to-set principle allows us to capture classical Hausdorff dimension through the pointwise analysis of the dimension of sets
- Let us use it to solve open problems in fractal geometry

References

- Jack H. Lutz and Neil Lutz, Algorithmic information, plane Kakeya sets, and conditional dimension, STACS 2017, ACM Transactions on Computation Theory (TOCT), to appear.
- Elvira Mayordomo, A point-to-set principle for separable metric spaces, in preparation

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- It can be done from measure
- For spaces with a suitable regularity condition it can be done through gambling (pretty useful for resource-bounds)
- Similarly **packing dimension** and **exact dimension** can be effectivized for all separable spaces

Exact dimension: Kolmogorov complexity characterization

Let g be increasing in both arguments. For $s \geq 0$,
$$H^{g,s}(E) = \lim_{\delta \to 0} \inf_{\mathcal{U} \text{ is a } \delta\text{-cover of } E} \sum_{U \in \mathcal{U}} g(s, \mathrm{diam}(U))$$

$$\dim_{\mathrm{H}}^{(g)}(E) = \inf \{ s \,|\, H^{g,s}(E) = 0 \} .$$

Definition

Let X be a separable metric space. Let $x \in X$,

$$\mathrm{cdim}_g^f(x) = \inf \left\{ s \,\middle|\, \exists^\infty r \ \mathrm{K}_r^f(x) \leq -\log(g(s, 2^{-r})) \right\} .$$

Theorem

Let $E \subseteq X$. Then

$$\dim_{\mathrm{H}}^{(g)}(E) = \min_{B \subseteq \{0,1\}^*} \mathrm{cdim}_g^{f,B}(E).$$