The Lovász local lemma and restrictions of Hindman’s theorem

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Joint work with Csima, Hirschfeldt, Jockusch, Solomon, and Westrick.

Hindman's finite sums theorem

Given $A \subseteq \mathbb{N}$, let $\mathrm{FS}(A)$ denote the set of all finite non-empty sums of elements of A.

Hindman's theorem (HT)**.** For every $k \geq 1$ and every $c : \mathbb{N} \to k$, there is an infinite set $H \subseteq \mathbb{N}$ such that c is constant on $\mathrm{FS}(H)$.

When we restrict HT to k-colorings for a specific k, we denote it by HT_k.

- Original proof by Hindman (1972), simplified by Baumgartner (1974).
- Ultrafilter proof by Galvin and Glazer (1977).
- Dynamics proof by Furstenburg and Weiss (1978).
- Reverse mathematics: Blass, Hirst, and Simpson (1987).
- A much simpler combinatorial proof by Towsner (2012).

Comparison with Ramsey's theorem

Given $A \subseteq \mathbb{N}$ and $n \geq 1$, let $[A]^n = \{(x_1, \ldots, x_n) \in A^n : x_1 < \cdots < x_n\}$.

A set $H \subseteq \mathbb{N}$ is homogeneous for $c : [\mathbb{N}]^n \to k$ if c is constant on $[H]^n$.

Ramsey's theorem (RT)**.** For all $n, k \geq 1$, every $c : [\mathbb{N}]^n \to k$ has an infinite homogeneous set.

RT^n_k denotes the restriction to a specific n and k.

There are also many proofs of RT, but many are quite elementary.

Example. How do you build 3-element solution to RT?

- Trivial for $n = 1$ and $n = 3$, not meaningful for $n > 3$.
- Given $c : [\omega]^2 \to 2$, how do you build a 3-element homogeneous set?

A 3-element solution to HT

Claim. Every $c : \mathbb{N} \to \{R, B\}$ is constant on $\mathrm{FS}(F)$ for some 3-element set F.

Proof. WLOG, say $c(0) = B$. We may assume $\exists^\infty x\, [c(x) = B]$.

If there exist positive $x < y$ with $c(x) = c(y) = c(x + y) = B$, take $F = \{0, x, y\}$. So assume not.

Fix $x_1 < x_2 < \cdots < x_6$ such that $c(x_i) = B$ for each i and the difference between any two consecutive x_i's is different.

Let $d_i = x_{i+1} - x_i$.

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6$

$d_1 \quad d_2 \quad d_3 \quad d_4 \quad d_5$

A 3-element solution to HT

$$\begin{array}{ccccccccccc} x_1 & & x_2 & & x_3 & & x_4 & & x_5 & & x_6 \\ & d_1 & & d_2 & & d_3 & & d_4 & & d_5 & \end{array}$$

By assumption, it must be that $c(d_i) = R$ for each i.

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By assumption, it must be that $c(d_i) = R$ for each i.

Similarly, the sum of any consecutive d_i's must also be colored R by c.

Finally, it cannot be that $c(d_1 + d_4) = c(d_2 + d_5) = c(d_1 + d_2 + d_4 + d_5) = B$.

So if $c(d_1 + d_4) = R$, we can take $F = \{d_1,\ d_2 + d_3,\ d_4\}$.

If $c(d_2 + d_5) = R$, we can take $F = \{d_2,\ d_3 + d_4,\ d_5\}$.

And if $c(d_1 + d_2 + d_4 + d_5) = R$, we can take $F = \{d_1 + d_2,\ d_3,\ d_4 + d_5\}$.

HT and reverse mathematics

Blass, Hirst, and Simpson (1987) proved that every computable instance of HT has a solution computable from $0^{(\omega+2)}$, but not necessarily $0'$.

Adapting Jockusch's results on RT^3_2, they showed that there is a computable instance all of whose solutions compute $0'$.

Theorem (Blass, Hirst, and Simpson, 1987)**.**

- HT is provable in ACA_0^+.
- Over RCA_0, HT_2 implies ACA_0.

Thirty years later, this is still the state of the art.

There has been quite a bit of work on extensions of HT.

Two restrictions

Given $A \subseteq \mathbb{N}$ and $n \geq 1$, let $\mathrm{FS}^{\leq n}(A)$ denote the set of all non-empty sums of at most n elements of A.

Let $\mathrm{HT}^{\leq n}$ and $\mathrm{HT}_k^{\leq n}$ denote the obvious restrictions of HT and HT_k.

Question (Hindman, Leader and Strauss, 2003). Is there a proof of $\mathrm{HT}^{\leq 2}$ that is not already a proof of the full HT?

From their paper: "It seems truly remarkable that this can be unknown."

Given $A \subseteq \mathbb{N}$ and $n \geq 1$, let $\mathrm{FS}^{=n}(A)$ denote the set of all sums of exactly n elements. Let $\mathrm{HT}^{=n}$ and $\mathrm{HT}_k^{=n}$ denote the obvious restrictions.

Obviously, $\mathrm{HT}_k \rightarrow \mathrm{HT}_k^{\leq n} \rightarrow \mathrm{HT}_k^{=n}$. Also, $\mathrm{RT}_k^n \rightarrow \mathrm{HT}_k^{=n}$.

HT for sums of length at most 2

A paradox:

- we know of no proof of $\mathsf{HT}_2^{\leq 2}$ other than the proof of the full HT,
- yet it is not at all clear how to show that $\mathsf{HT}_2^{\leq 2}$ is not computbaly true.

Recall that a coloring $c : [\mathbb{N}]^2 \to 2$ is stable if $(\forall x) \lim_y f(x, y)$ exists.

SRT_2^2 is the restriction of Ramsey's theorem to stable colorings.

Theorem (Dzhafarov, Jockusch, Solomon, and Westrick)**.**

Over RCA_0, $\mathsf{HT}_2^{\leq 2}$ implies SRT_2^2.

Thus, in particular, there is a computable instance of $\mathsf{HT}_2^{\leq 2}$ with no computable solution.

Apartness

Fix $b \geq 2$ and $x \in \mathbb{N}$. If $x = i_0 \cdot b^{e_0} + \cdots + i_t \cdot b^{e_t}$ where $i_0, \ldots, i_t \in \{1, \ldots, b-1\}$ and $e_0 < \cdots < e_t$, let $\lambda_b(x) = e_0$ and $\mu_b(x) = e_t$.

Say two natural numbers $x < y$ are <u>b-apart</u> if $\mu_b(x) < \lambda_b(y)$.

HT **with b-apartness** is the statement of HT in which all elements of the monochromatic are required to be pairwise b-apart.

Facts.

- For each k, $b \geq 2$, RCA_0 proves $\mathrm{HT}_k \leftrightarrow \mathrm{HT}_k$ with b-apartness.
- For each $b \geq 2$, RCA_0 proves $\mathrm{HT} \leftrightarrow \mathrm{HT}$ with b-apartness.

In fact, all of these are strong computable equivalences.

The proof that HT implies HT with b-apartness does not lift to also show $\mathrm{HT}^{\leq n}$ with b-apartness implies $\mathrm{HT}^{\leq n}$ with b-apartness.

HT with apartness

Theorem (Carlucci, Kołodziejczyk, Lepore, and Zdanowski, 2017)**.**

- For any $b \geq 2$, RCA_0 proves that $\mathsf{HT}_2^{\leq 2}$ with b-apartness implies ACA_0.
- RCA_0 proves that $\mathsf{HT}_4^{\leq 2}$ implies ACA_0.

The apartness condition is not really "cheating". It is used in most proofs of/from Hindman's theorem, and was present in the original formulation. It can also be recast as a natural principle, the Finite unions theorem.

Corollary. Our best bounds for $\mathsf{HT}^{\leq 2}$ are the same as for the full HT.

A note on strong reductions

- Our proof that $\mathsf{HT}_2^{\leq 2} \to \mathsf{SRT}_2^2$ actually shows that $\mathsf{SRT}_2^2 \leq_{\mathrm{sc}} \mathsf{HT}_2^{\leq 2}$.
- Carlucci (2017) showed that $\mathsf{IPT}_2^2 \leq_{\mathrm{sc}} \mathsf{HT}_4^{\leq 2}$, where IPT_2^2 is the strictly stronger increasing polarized Ramsey's theorem for pairs.

HT for sums of length exactly 2

$\mathsf{HT}^{=n}_k$ is an obvious corollary of RT^n_k.

Theorem (Carlucci, Kołodziejczyk, Lepore, and Zdanowski, 2017)**.**

If $n|m$ then $\mathsf{HT}^n \leq_{\mathrm{sc}} \mathsf{HT}^m$.

<u>Proof.</u>

Fix $c : \mathbb{N} \to k$. Say $m = nd$. Let $H = \{x_1 < x_2 < \cdots\}$ be an infinite set such that c is constant on $\mathsf{FS}^{=m}(H)$. Now define G to be the set $\{x_1 + \cdots + x_d,\ x_{d+1} + \cdots + x_{2d+1},\ \ldots\}$. Then c is constant on $\mathsf{FS}^{=n}(G)$.

Theorem (Carlucci, Kołodziejczyk, Lepore, and Zdanowski, 2017)**.**

For any $n \geq 3$, $b \geq 2$, $\mathsf{HT}^{=n}$ with b-apartness is equivalent to ACA_0.

What about $\mathsf{HT}^{=2}$? Can we at least show it's not computably true?

Diagonalization strategy

We want to build a computable coloring $c : \mathbb{N} \to 2$.

For each e, wait for a certain-sized finite $F_e \subseteq W_e$ to be enumerated.

For sufficiently large s, ensure $F_e + s$ is not homogeneous.

Dealing with a single c.e. set W.

- Wait for some $x < y$ in W to be enumerated into W. Let $d = y - x$.
- For each $s \leq d$ let $c(s) = 0$.
- For $s > d$, having inductively defined $c \upharpoonright s$, define $c(s) = 1 - c(s - d)$.
- Now $c(y + s) = 1 - c(y + s - d) = 1 - c(x + s)$ for all large enough s.

Diagonalization strategy

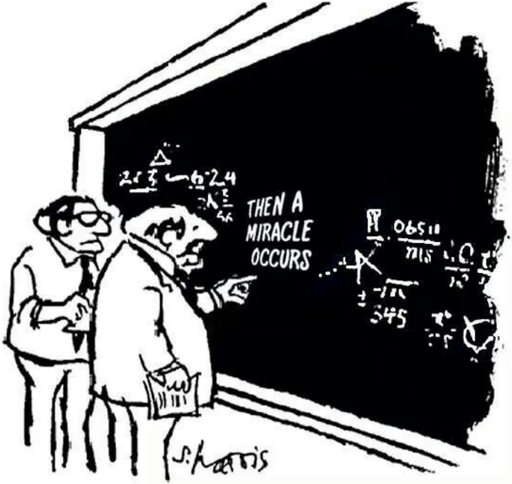
The basic strategy fails even for two c.e. sets, W_0 and W_1.

<u>Example.</u>

- Suppose $F_0 = \{0, 1\}$ and $F_1 = \{0, 2\}$.
- Then for all s, one of $F_0 + s$, $F_1 + s$, $F_0 + (s + 1)$ must be homogeneous.

This failure gives us some insights.

- The probability that $F_e + s$ is homogeneous is only $2^{-|F_e|+1}$.
- If $s < t$ are far enough apart, then $F_e + s$ and $F_i + t$ are disjoint.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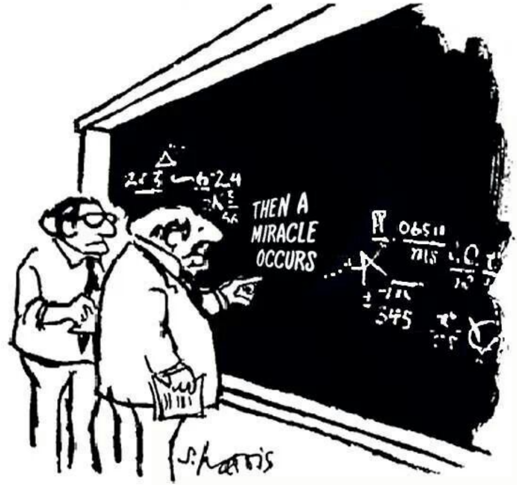

* Thanks to Jason Bell and Jeff Shallit (U Waterloo).

An application of the Lovász local lemma

Consider a collection $x_0, x_1, \ldots$ of independent binary random variables.

A clause is a finite sequence $x_{n_0} = i_0 \vee \cdots \vee x_{n_k} = i_k$, where $i_0, \ldots, i_k \in \{0, 1\}$.

A CNF is an infinite conjunction of clauses.

A satisfying assignment for a CNF is a map $c : \mathbb{N} \to \{0, 1\}$ such that each conjunct in the CNF has a disjunct $x_n = i$ and $c(n) = i$.

Theorem (Rumyantsev and Shen, 2014).

For every $\alpha \in (0, 1)$, there exists an $N \in \mathbb{N}$ such that every computable infinite CNF in which all clauses have size at least N, and for all $m \geq N$, every variable appears in at most $2^{\alpha m}$ clauses of size m, has a computable satisfying assignment.

An application of the Lovász local lemma

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For every $\alpha \in (0, 1)$, there exists an $N \in \mathbb{N}$ such that every computable infinite CNF in which all clauses have size at least N, and for all $m \geq N$, every variable appears in at most $2^{\alpha m}$ clauses of size m, has a computable satisfying assignment.

Let $\alpha = 0.5$. Fix N as above. For each e, wait for $F_e \subseteq W_e$ of size $N + e$.

Take the CNF whose clauses are $\bigvee_{n \in F_e + s} x_n = 0$ and $\bigvee_{n \in F_e + s} x_n = 1$ for all sufficiently large s.

If c is a satisfying assignment and W_e is infinite, then c is not homogeneous on $F_e + s$ for all sufficiently large s.

Corollaries

Theorem (Csima, D., Hirschfeldt, Jockusch, Solomon, and Westrick)**.**

There exists a computable instance of $\mathsf{HT}_2^{=2}$ with no computable solution.

Corollary. RCA_0 does not prove $\mathsf{HT}_2^{=2}$.

A modification of the argument also yields the following:

Theorem (Csima, D., Hirschfeldt, Jockusch, Solomon, and Westrick)**.**

There exists a computable instance of $\mathsf{HT}_2^{=2}$ every solution of which computes a $\mathsf{DNC}(0')$ function.

Corollary. RCA_0 proves $\mathsf{HT}_2^{=2} \to \mathsf{RRT}_2^2$.

Here, RRT_2^2 is the Rainbow Ramsey's theorem for pairs.

Ramseyan factorization theorem

Murakami, Yamazaki, and Yokoyama introduced the following principle in connection with their work on the Ramseyan factorization theorem.

Fix $n, k \geq 1$ and $f : [\mathbb{N}]^n \to \mathbb{N}$.

RT^f_k is the statement that for every $c : \mathbb{N} \to k$ there is an infinite set $H \subseteq \mathbb{N}$ such that $c \circ f$ is constant on $[H]^n$.

If $f(x_1, \ldots, x_n) = x_1 + \cdots + x_n$ for all $x_1, \ldots, x_n \in \mathbb{N}$ then $\mathsf{RT}^f_k = \mathsf{HT}^{=n}_k$.

Theorem (Murakami, Yamazaki, and Yokoyama, 2014)**.**

- RCA_0 proves $\mathsf{RT}^n_k \to (\forall f : [\mathbb{N}]^n \to \mathbb{N})\ \mathsf{RT}^f_k$.
- If $f : [\mathbb{N}]^n \to \mathbb{N}$ is a bijection then $\mathsf{RT}^f_k \leftrightarrow \mathsf{RT}^n_k$ over RCA_0.

Addition-like functions

A computable function $f : [\mathbb{N}]^2 \to \mathbb{N}$ is addition-like if

- there is a computable function g such that $y > g(x, n) \to f(x, y) > n$,
- there is a b such that $|\{y : f(x, y) = k\}| < b$ for all $x, k \in \mathbb{N}$.

Examples.

- Addition.
- Subtraction/difference.

Theorem (Csima, D., Hirschfeldt, Jockusch, Solomon, and Westrick).

For each addition-like f, there exists a computable instance of RT^f_2 all of whose solutions compute a $\mathrm{DNC}(0')$ function.

Corollary. For each addition-like f, RCA_0 proves $\mathrm{RT}^f_2 \to \mathrm{RRT}^2_2$.

Further applications

Theorem (Cholak, D., Hirschfeldt, and Patey)**.**

There exists an instance of $\mathsf{HT}_2^{=2}$ such that the class of oracles that compute a solution to c has measure 0.

OVW(2, 2) is the Ordered variable word problem for 2-element alphabets.

Miller and Solomon (2004) constructed a computable instance of OVW(2, 2) with no computable solution.

Theorem (Liu, Monin, and Patey, 2018)**.**

There exists a computable instance of OVW(2, 2) all of whose solutions compute a DNC($0'$) function.

Thanks for your attention!