Dense Computability, Upper Cones, and Minimal Pairs

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Joint work with Eric P. Astor and Carl G. Jockusch, Jr.

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A set S has **density 0** if $\limsup_n \frac{|S \cap [0, n)|}{n} = 0$.

For a description Δ of A, let

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- $M = \{n : \Delta(n)\downarrow = 1 - A(n)\}$, and
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- a **generic description** of A if $M \cup R = \emptyset$.
- a **coarse description** of A if $D \cup R = \emptyset$.
- an **effective dense description** of A if $D \cup M = \emptyset$.

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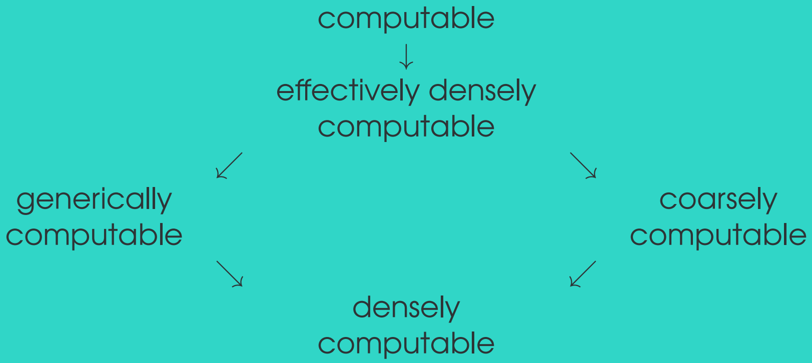
These notions can be relativized to define dense computability relative to X, etc.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We can similarly define **generic reducibility** $\leqslant_g$, **dense reducibility** $\leqslant_d$, and **effective dense reducibility** $\leqslant_{ed}$.

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Cor (Kautz). If $Y >_{\mathbf{T}} \emptyset$ and X is weakly 2-random relative to Y then X, Y form a minimal pair.

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Open Question. Is there a minimal pair for generic reducibility?

Open Question. Is there a minimal pair for effective dense reducibility?