Scott Sentences of Finitely-Generated Groups

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A question

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- Finite groups can be characterized by a single first-order sentence.
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How hard is it to describe a group up to isomorphism?

- Finite groups can be characterized by a single first-order sentence.
- $\aleph_0$-categorical groups can be characterized by its first-order theory within countable groups.
We will work in $L_{\omega_1,\omega}$, where we allow countable conjunctions and countable disjunctions.

Theorem (Scott, '65)

For every countable structure in a countable language, there's a sentence whose countable models are exactly the isomorphic copies of the structure. Such a sentence is called a Scott sentence.

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We are interested in finding "optimal" Scott sentences for finitely-generated groups.
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- We are interested in finding “optimal” Scott sentences for finitely-generated groups.
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- For a given structure, the complexity of a computable Scott sentence is higher than or equal to the complexity of the index set.
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H. (‘17): Free nilpotent groups of infinite rank, polycyclic groups, lamplighter groups, solvable Baumslag-Solitar groups, (Gromov) random groups
Theorem (Knight, Saraph)

Every computable finitely-generated group has a $\Sigma_3$ computable Scott sentence.
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For $G = \langle a \rangle$, consider $\exists x (\forall y \exists w \ w(x) = y) \land (\forall r \ r(a) \leftrightarrow r(x))$. 

Finitely-generated free groups, infinite dihedral groups, polycyclic groups, lamplighter groups, solvable groups, and random groups all have a computable $d$-$\Sigma_2$ Scott sentence.

For $\mathbb{Z}$, consider $(\exists x \forall y \forall k \ k \neq x) \land (\forall x (\forall y \forall k \ k \neq x) \rightarrow (\forall y \exists k \ y = kx))$.
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Question (Knight, Saraph)

Does every finitely-generated computable group have a computable d-$\Sigma_2$ Scott sentence?
Main Lemma

\[(\exists x \forall y \forall k \, ky \neq x) \land \forall x(\forall y \forall k \, ky \neq x) \rightarrow (\forall y \exists k \, y = kx)\]
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Theorem (Alvir, Knight, McCoy)

Let \(A\) be a computable finitely-generated structure. Then the following are equivalent:

1. \(A\) has a computable \(d\-\Sigma_2\) Scott sentence.
2. The orbit of some (equivalently, all) generating tuple is defined by a computable \(\Pi_1\) formula.

Theorem (Harrison-Trainor, H.)

Let \(A\) be a finitely-generated structure. Then \(A\) has no \(d\-\Sigma_2\) Scott sentence if and only if \(A\) is self-reflective, i.e. \(A\) has a proper substructure \(B\) such that \(A \sim B\) and \(B \leq A\).
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Let A be a finitely-generated structure. Then A has no d-\(\Sigma_2\) Scott sentence if and only if A is self-reflective, i.e. A has a proper substructure B such that \(A \cong B\) and \(B \leq_1 A\).
A finitely-generated structure $A$ is *self-reflective* if $A$ has a proper substructure $B$ such that $A \cong B$ and $B \leq_1 A$. 

Theorem (Harrison-Trainor, H.) There is a computable self-reflective group. Thus, it does not have a $d\Sigma_2$ Scott sentence. 

We first construct a computable finitely-generated structure that is self-reflective. Then we use small cancellation theory to code the structure into a computable self-reflective finitely-generated group.
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Constructing the example

A finitely-generated structure $A$ is \textit{self-reflective} if $A$ has a proper substructure $B$ such that $A \cong B$ and $B \leq_1 A$.

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**Question**

*Does every finitely-presented computable group have a (computable) $d$-$\Sigma_2$ Scott sentence?*


Thank you!