Scott Sentences of Finitely-Generated Groups

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A question

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How hard is it to describe a group up to isomorphism?

- Finite groups can be characterized by a single first-order sentence.
- $\aleph_0$-categorical groups can be characterized by its first-order theory within countable groups.

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- We will be working with the quantifier complexity classes (Σ_n, Π_n, and d-Σ_n formulas.)
- We are interested in finding "optimal" Scott sentences for finitely-generated groups.

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- We will be working with the arithmetical hierarchy (Σ_n, Π_n, and d-Σ_n sets) and m-degrees.
- For a given structure, the complexity of a computable Scott sentence is higher than or equal to the complexity of the index set.

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- Calvert, Harizanov, Knight, Miller (‘06): Vector spaces, Archimedean ordered fields, reduced abelian p-groups

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- H. (‘17): Free nilpotent groups of infinite rank, polycyclic groups, lamplighter groups, solvable Baumslag-Solitar groups, (Gromov) random groups

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For $\mathbb{Z}$, consider $(\exists x \forall y \forall k \; ky \neq x) \wedge \forall x (\forall y \forall k \; ky \neq x) \rightarrow (\forall y \exists k \; y = kx)$.

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Question (Knight, Saraph)

Does every finitely-generated computable group have a computable d-Σ_2 Scott sentence?

Main Lemma

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Theorem (Alvir, Knight, McCoy)

Let A be a computable finitely-generated structure. Then the following are equivalent:

1. *A has a computable d-Σ_2 Scott sentence.*
2. *The orbit of some (equivalently, all) generating tuple is defined by a computable Π_1 formula.*

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Theorem (Harrison-Trainor, H.)

Let A be a finitely-generated structure. Then A has no d-Σ_2 Scott sentence if and only if A is self-reflective, i.e. A has a proper substructure B such that $A \cong B$ and $B \leq_1 A$.

Constructing the example

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Then we use small cancellation theory to code the structure into a computable self-reflective finitely-generated group.

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Question

Does every finitely-presented computable group have a (computable) d-Σ_2 Scott sentence?

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Thank you

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