Permutations of the integers do not induce nontrivial automorphisms of the Turing degrees

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June 5, 2018

- Part I: Structure of the Turing degrees (2002)
- Part II: Randomness for Bernoulli measures (2009)
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- Part I: Structure of the Turing degrees

The Turing degrees

Definition 1

For $A, B \subseteq \omega$, we say that $A \leq_T B$ if A can be computed from B. If $A \leq_T B$ and $B \leq_T A$ then we say that $A \equiv_T B$. If $\mathbf{a} = \{X \mid X \equiv_T A\}$ denotes the Turing degree of A and we define $\mathbf{a} \leq \mathbf{b} \leftrightarrow A \leq_T B$, then $\mathcal{D} = \{\mathbf{a} \mid A \subseteq \omega\}$ is the upper semilattice of Turing degrees.

L.u.b., not always g.l.b.

The least upper bound of two tasks in terms of difficulty should be to do them both.

The greatest lower bound – it is not so clear what that would be. This is reflected in the fact that the Turing degrees form an upper semilattice which is not a lattice, i.e. any two degrees have a l.u.b. but some do not have a g.l.b.

Here the task corresponding to a degree $\mathbf{a}$ is to answer questions of the form "$x \in A$?".

Automorphisms, subalgebras, ideals

Note that 0 and $\leq$ are definable from the l.u.b. operation $\vee$. Shore and Slaman (1999) showed that so is the map $\mathbf{a} \mapsto \mathbf{a}'$. Here are a few natural things we should know about the algebra $\langle \mathcal{D}, \vee \rangle$, in order to say that we understand it.

Automorphisms - bijections $\pi : \mathcal{D} \to \mathcal{D}$ such that
$$\pi(\mathbf{x} \vee \mathbf{y}) = \pi(\mathbf{x}) \vee \pi(\mathbf{y}).$$

Subalgebras - subsets of $\mathcal{D}$ closed under $\vee$.

Ideals - sets $\mathcal{I} \subseteq \mathcal{D}$ that are “ideal elements” in the sense that if we write $\mathbf{a} \leq \mathcal{I}$ for $\mathbf{a} \in \mathcal{I}$, then
$\mathbf{a} \leq \mathbf{b} \leq \mathcal{I} \Rightarrow \mathbf{a} \leq I$, and
$\mathbf{a} \leq \mathcal{I}, \mathbf{b} \leq \mathcal{I} \Rightarrow \mathbf{a} \vee \mathbf{b} \leq \mathcal{I}$.

Groszek and Slaman (1983) showed that isomorphism types of subalgebras and ideals are not determined by ZFC. Open problem: Does $\langle \mathcal{D}, \vee \rangle$ have any nontrivial automorphism?

History of ideals

Given a cardinal κ, let $P(\kappa) \Leftrightarrow$ Ideals of $\mathcal{D}$ size $\leq \kappa$ realize all conceivable isomorphism types.

- '54 Kleene-Post $\vdash P(1)$ (trivial)
- '56 **Spector** $\vdash P(2)$ (forcing with trees)
- '65 Titgemeyer $\vdash P(3)$
- '68 Lachlan $\vdash P(4)$
- '69 Lerman $\vdash P(5)$
- '71 Lerman $\vdash (\forall \kappa < \aleph_0) P(\kappa)$ (homogeneity)
- '76 **Lachlan-Lebeuf** $\vdash P(\aleph_0)$ (embeddings)
- '83 Groszek-Slaman: ZFC $\nvdash P(2^{\aleph_0})$
- '86 Abraham-Shore $\vdash P(\aleph_1)$

Σ_1-presentable semilattices

We say that an upper semilattice L is Σ^0_1-presentable if there exists a transitive and reflexive Σ^0_1 relation $\leq$ on ω and a Δ^0_1 binary operation $\vee$ on ω such that if we mod out by $a \equiv b \Leftrightarrow a \leq b \,\&\, b \leq a$ then we get L with its order and l.u.b. operation. Similarly we get a notion of $\Sigma^0_1(\mathbf{a})$-presentable for any Turing degree $\mathbf{a}$.

Lemma 2

$[\mathbf{a}, \mathbf{b}]$ *is* $\Sigma^0_3(\mathbf{b})$*-presentable, for any Turing degrees* $\mathbf{a} \leq \mathbf{b}$*.*

Proof sketch: $\mathbf{a} \leq \mathbf{b}$ if there exists e such that for all x, $\{e\}^B(x)$ halts and equals $A(x)$. A representative of the Turing degree $\mathbf{a} \vee \mathbf{b}$ is given by $A \oplus B := \{2x \,|\, x \in A\} \cup \{2x+1 \,|\, x \in B\}$. And $\{e \,|\, \{e\}^B$ is total$\}$ is computable from B''. []
For many $\mathbf{a}$, $\mathbf{b}$ this is also best possible (e.g. for $\mathbf{b} \geq \mathbf{0}'$). So in describing $\mathcal{D}$ we seem to have to use $\mathcal{D}$ itself!

Some useful tools

Theorem 3 (Friedberg 1957)

If $\mathbf{x} \geq \mathbf{0}'$ then there exists $\mathbf{a}$ with $\mathbf{a}' = \mathbf{x}$.

Theorem 4 (Jockusch, Posner 1978)

If $\mathbf{a} \leq \mathbf{0}'$ and $[\mathbf{0}, \mathbf{a}]$ is a lattice then $\mathbf{a}'' = \mathbf{0}''$ and hence $[\mathbf{0}, \mathbf{a}]$ is Σ^0_3-presentable.

Application to automorphisms

Theorem 5

Let $\mathbf{a} \geq \mathbf{0}''$ and let L be a lattice with 0 and 1. The following are equivalent:

1. L *is* $\Sigma^0_1(\mathbf{a})$*-presentable.*
2. $L \cong [\mathbf{0}, \mathbf{g}]$ *for some* $\mathbf{g}'' \leq \mathbf{a}$*.*

But the collection of $\Sigma^0_1(\mathbf{a})$-presentable L determines $\mathbf{a}$. This gives a “reason” why the following result is true:

Corollary 6 (Slaman, Woodin)

Every automorphism of $\langle \mathcal{D}, \leq, ' \rangle$ *is the identity above* $\mathbf{0}''$*.*

- Part II: Randomness for Bernoulli measures

Randomness

Definition

A test for μ-randomness is a uniformly $\Sigma_1^0(\mu)$ sequence $\{U_n\}_{n\in\omega}$ with $\mu(U_n) \leq 2^{-n}$.

If X passes all μ-randomness tests then X is μ-*random*.

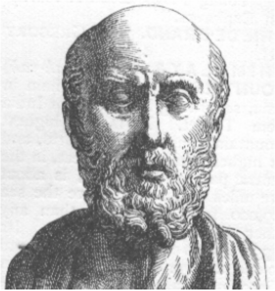
Martin-Löf randomness for arbitrary measures on 2^ω (Reimann and Slaman).

Hippocratic randomness

Definition

A test for Hippocratic μ-randomness is a uniformly Σ^0_1 sequence $\{U_n\}_{n\in\omega}$ with $\mu(U_n) \le 2^{-n}$.

If X passes all Hippocratic tests then X is *Hippocrates μ-random*.



Like Hippocrates we are not consulting the oracle μ but rather looking for "natural causes".

Bernoulli measures

For each $n \in \omega$,

$$\mu_p(\{X : X(n) = 1\}) = p$$

$$\mu_p(\{X : X(n) = 0\}) = 1 - p$$

and $X(0), X(1), X(2), \ldots$ are mutually independent random variables.

Proposition

Consider an i.i.d. sequence $Y = \{Y_n\}_{n\in\omega}$ of Bernoulli(p) random variables, and the sample average $\overline{Y}_n = \frac{1}{n}\sum_{i=0}^{n-1} Y_i$. Let $N(b) = 2^{3b-1}$ and let

$$U_d = \bigcup_{b\geq d}\{Y : |\overline{Y}_{N(b)} - p| \geq 2^{-b}\}.$$

Then U_d is uniformly $\Sigma^0_1(p)$, and $\mu_p(U_d) \leq 2^{-d}$, i.e., $\{U_d\}_{d\in\omega}$ is a μ_p-ML-test.

The idea of the proof is to use Chebyshev's inequality and the fact that the variance of a Bernoulli(p) random variable is bounded (by $1/4$).

The essence of Statistics

Theorem

If Y is μ_p-random then $Y \geq_T p$.

Proof.

Let $\{U_d\}_{d\in\omega}$ be as in Proposition 15. Since Y is μ_p-random, $Y \notin \cap_d U_d$, so fix d with $Y \notin U_d$. Then for all $b \geq d$, we have

$$|\overline{Y}_{N(b)} - p| < 2^{-b}$$

Therefore, p is computable from Y. □

Theorem

There is a Hippocratic μ_p-test such that if Y passes this test then Y computes an accumulation point q of $\{\overline{Y}_n\}_{n\in\omega}$.

Proof.

Let

$$V_d := \{Y : \exists a, b \geq d \ |\overline{Y}_{N(a)} - \overline{Y}_{N(b)}| \geq 2^{-a} + 2^{-b}\}$$

Then $\{V_d\}_{d \in \omega}$ is uniformly Σ^0_1. Recall

$$U_d = \{Y : \exists b \geq d \ |\overline{Y}_{N(b)} - p| \geq 2^{-b}\}$$

We have $V_d \subseteq U_d$. [1]
Therefore $\mu_p(V_d) \leq \mu_p(U_d) \leq 2^{-d}$ for all p. Thus if Y is Hippocrates μ_p-random then $Y \notin V_d$ for some d. This allows Y to compute the limit q of the sequence $\{Y_{N(b)}\}_{b \geq d}$.

□

[1] (If $|\overline{Y}_{N(b)} - p| < 2^{-b}$ for all $b \geq d$ then $|\overline{Y}_{N(a)} - \overline{Y}_{N(b)}| \leq |\overline{Y}_{N(a)} - p| + |p - \overline{Y}_{N(b)}| < 2^{-a} + 2^{-b}$ for all $a, b \geq d$.)

Theorem

If Y is Hippocrates μ_p-random then Y satisfies the Strong Law of Large Numbers for p.

Proof.

Let q_1, q_2 be rational numbers with $q_1 < p < q_2$. Let

$$W_N := \{Y : \exists n \geq N\ \overline{Y}_n \leq q_1\} \cup \{Y : \exists n \geq N\ \overline{Y}_n \geq q_2\}$$

Then $\{W_N\}_{N \in \omega}$ is uniformly Σ^0_1, and $\mu_p W_N \to 0$ effectively:

$$\mu_p = \{Y : \exists n \geq N\ \overline{Y}_n \leq q_1\} \leq \frac{1}{2|p - q_1|} \sum_{n \geq N} \frac{3}{n^2} - \frac{2}{n^3}$$

□

Theorem

If Y is Hippocrates μ_p-random then $Y \geq_T p$.

Proof.

By Theorem 17, Y computes the limit of a subsequence $\{Y_{N(b)}\}_{b\in\omega}$. By Theorem 19, this limit must be p. $\square$

Theorem

For all p, if there is a Hippocratic μ_p-test $\{U_n\}_{n\in\omega}$ such that $\{X : X \not\geq_T p\} \subseteq \cap_n U_n$, then p is computable.

Proof.

Let $\{U_n\}_{n\in\omega}$ be such a test. U_1^c has a low member X_1 and a hyperimmune-free member X_2. By assumption $X_1 \geq_T p$ and $X_2 \geq_T p$, so p is both low and hyperimmune-free, hence p is computable. □

Corollary

There is no universal Hippocratic μ_p-test (unless p is computable).

Turning a $\Sigma^0_1(p)$ test into a Σ^0_1 test

Definition

Let $\{O_n\}_{n\in\omega}$ be a universal μ_p-test for all p, i.e. $\mu_p O^p_n \leq 2^{-(n)}$ for all p and $\{(p, X, n) : X \in O^p_n\}$ is Σ^0_1.

Definition

Let Ψ_d denote the reduction from Theorem 17 under the assumption $Y \notin U_d$ there.

Theorem

If Y is Hippocrates μ_p-random then Y is μ_p-random.

Proof. We have

$$\{Y : Y \notin U_d\} \subseteq \{Y : \Psi_d^Y = p\}$$

$$\cup\{Y : Y \text{ not Hippocrates } \mu_p\text{-random.}\}$$

Let

$$V_n^{(d)} := \left\{X : \exists k \ \left(\Psi_d^X \upharpoonright k \downarrow \& \ \ X \in O_n^{\Psi_d^X \upharpoonright k}\right)\right\}$$

Then

$$V_n^{(d)} \subseteq O_n^p \cup \{Y : \Psi_d^Y \neq p\}$$

So

$$\mu_p(V_n^{(d)}) \leq \mu_p(O_n^p) + \mu_p U_d \leq 2^{-n} + 2^{-d}$$

Form the diagonal $W_n = V_n^{(n)}$; then $\{W_n\}_{n\in\omega}$ is a Hippocratic μ_p-test.
Suppose for contradiction that Y is Hippocrates μ_p-random but not μ_p-random. Then for all n, $Y \in O_n^p$. Fix d such that $Y \notin U_d$, so $\Psi_d^Y = p$. Then for all $n \geq d$, $\Psi_n^Y = p$. Then $Y \in \cap_{n\geq d} V_n^{(n)}$. So Y is not Hippocrates μ_p-random. []

- Part III: Permutations don't induce automorphisms of Aut($\mathcal{D}$)

The Turing degrees

Two approaches:

- $\mathcal{D}_T = \omega^\omega / \equiv_T$
- $\mathcal{D}_T = 2^\omega / \equiv_T$

Same abstract structure, different notions of “inducing”.

Open problem

Question

Does $(\mathcal{D}_T, \leq)$ *have any nontrivial automorphisms?*

- $\pi : \mathcal{D}_T \to \mathcal{D}_T$ is an automorphism if it is bijective and $\mathbf{x} \leq \mathbf{y} \iff \pi(\mathbf{x}) \leq \pi(\mathbf{y})$.
- $\pi : \mathcal{D}_T \to \mathcal{D}_T$ is *nontrivial* if $(\exists \mathbf{x})(\pi(\mathbf{x}) \neq \mathbf{x})$.

Other degree structures

- The hyperdegrees $\mathcal{D}_h$ have no nontrivial automorphisms (Slaman, Woodin $\sim$1990).
- The Turing degrees $\mathcal{D}_T$ have at most countably many (Slaman, Woodin $\sim$1990).
- The many-one degrees $\mathcal{D}_m$ have many automorphisms.

History of Aut(D_T).

1980 Nerode and Shore show each automorphism equals the identity on some cone.

1990 Slaman and Woodin announce and circulate proofs that the cone can be lowered to $0''$, and Aut(D_T) is countable.

1999 Cooper sketches a construction of a nontrivial automorphism, but does not finish that project. Proposed automorphism π is induced by a continuous map on ω^ω.

2008 Outline of Slaman-Woodin results published.

2015 No automorphisms induced by permutations.

Inducing, from ω to 2^ω to $\mathcal{D}_T$

Definition

The pullback of $f : \omega \to \omega$ *is* $f^* : \omega^\omega \to \omega^\omega$ *given by*

$$f^*(A)(n) = A(f(n)).$$

We often write $F = f^*$*.*

$$\pi([A]_T) = [F(A)]_T$$

Plausible that a permutation would induce an automorphism?

Theorem (Haught and Slaman 1993)

A permutation of ω (actually $2^{<\omega}$) can induce an automorphism of

$$(\mathit{PTIME}^A, \leq_{pT}).$$

Caveat: the automorphism is probably not in the ideal itself.

Plausible that a permutation would induce an automorphism?

Theorem (Kent $\sim$1967)

There exists a permutation f such that

(i) *for all recursively enumerable B, $f(B)$ and $f^{-1}(B)$ are recursively enumerable (and hence for all recursive A, $f(A)$ and $f^{-1}(A)$ are recursive);*

(ii) *f is not recursive.*

So a noncomputable f may map the Turing degree $\mathbf{0}$ to $\mathbf{0}$.

Definition

$A \subset \omega$ is cohesive if for each recursively enumerable set W_e, either $A \cap W_e$ is finite or $A \cap (\omega \setminus W_e)$ is finite.

Proof.

Kent's permutation is just any permutation of a cohesive set (and the identity off the cohesive set). $\square$

The case $D_T = \omega^\omega / \equiv_T$ is trivial

$$f^*(f^{-1})(n) = f^{-1}(f(n)) = n$$

so

$$f^{-1} \mapsto_{f^*} \text{id}_\omega$$

$\therefore f^*$ maps f^{-1} to a computable function $\therefore f^{-1}$ is computable $\therefore$ f is computable

For 2^ω, one idea is: think of the elements of 2^ω as probabilities.

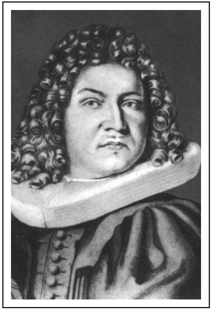
Bernoulli measures

For each $n \in \omega$,

$$\mu_p(\{X : X(n) = 1\}) = p$$

$$\mu_p(\{X : X(n) = 0\}) = 1-p$$

and
$X(0), X(1), X(2), \ldots$
are mutually independent random variables.



Jakob Bernoulli

Lebesgue Density

Ben Miller (2008) proved an extension of the Lebesgue Density Theorem to Bernoulli measures and beyond.

Definition

An ultrametric space is a metric space with metric d satisfying the strong triangle inequality

$$d(x,y) \leq \max\{d(x,z), d(z,y)\}.$$

Lebesgue Density

Definition

A Polish space is a separable completely metrizable topological space.

Definition

In a metric space, $B(x,\varepsilon) = \{y : d(x,y) < \varepsilon\}$.

Lebesgue Density

Theorem (Lebesgue Density Theorem for a class including μ_p on 2^ω)

Suppose that X is a Polish ultrametric space, μ is a probability measure on X, and $\mathcal{A} \subseteq X$ is Borel. Then $\lim_{\varepsilon \to 0} \frac{\mu(\mathcal{A} \cap B(x,\varepsilon))}{\mu(B(x,\varepsilon))} = 1$ for μ-almost every $x \in \mathcal{A}$.

Lebesgue Density

Definition

For any measure μ *define the conditional measure by*

$$\mu(\mathcal{A} \mid \mathcal{B}) = \frac{\mu(\mathcal{A} \cap \mathcal{B})}{\mu(\mathcal{B})}.$$

A measurable set $\mathcal{A}$ *has density* d *at* X *if*

$$\lim_n \mu_p(\mathcal{A} \mid [X \upharpoonright n]) = d.$$

Lebesgue Density

Let $\Xi(\mathcal{A}) = \{X : \mathcal{A}$ has density 1 at $X\}$.

Corollary (Lebesgue Density Theorem for μ_p)

For Cantor space with Bernoulli(p) product measure μ_p, the Lebesgue Density Theorem holds:

$$\lim_{n\to\infty} \frac{\mu_p(\mathcal{A} \cap [x \restriction n])}{\mu_p([x \restriction n])} = 1$$

for μ-almost every $x \in \mathcal{A}$.
If $\mathcal{A}$ is measurable then so is $\Xi(\mathcal{A})$. Furthermore, the measure of the symmetric difference of $\mathcal{A}$ and $\Xi(\mathcal{A})$ is zero, so $\mu(\Xi(\mathcal{A})) = \mu(\mathcal{A})$.

Lebesgue Density

Proof.

Consider the ultrametric $d(x, y) = 2^{-\min\{n : x(n) \neq y(n)\}}$. It induces the standard topology on 2^ω. □

Law of the Iterated Logarithm

Theorem (Khintchine 1924)

Let Y_n be independent, identically distributed random variables with means zero and unit variances. Let $S_n = Y_1 + \dots Y_n$. Then

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{n \log\log n}} = \sqrt{2}, \qquad \textit{a.s.},$$

where $\log$ is the natural logarithm, $\limsup$ denotes the limit superior, and "a.s." stands for "almost surely".

Corollary (Kjos-Hanssen 2010)

Each μ_p-random computes p (layerwise!).

The idea now is that the permutation f of ω preserves something, namely μ_p for any p.

Main theorem

Theorem

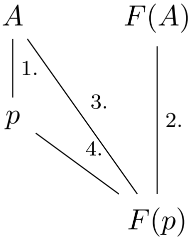
A permutation $f : \omega \to \omega$ induces an automorphism of $\mathcal{D}_T$ iff f is computable.

Two proof steps.

First show f induces the trivial automorphism. Then use that to show f is computable. □

Steps of the proof

Assume A is F-μ_p-ML-random.



1. $p \leq_T A$ (Law of the Iterated Logarithm)
2. $F(p) \leq_T F(A)$
3. $F(p) \leq_T A$
4. $F(p) \leq_T p$ (Lebesgue Density Theorem & Sacks/de Leeuw, Moore, Shannon, Shapiro)

Majority vote computation of F

If F induces the trivial automorphism of $\mathcal{D}_T$, we prove F is computable.
Notation: $A + n = A \cup \{n\}$, $A - n = A \setminus \{n\}$.
We use Lebesgue Density again, this time for $p = 1/2$.

We have $F(A) \leq_T A$. Fix Φ which works for $1 - \frac{\varepsilon}{2}$ measure many A.

$$\begin{array}{ccc} F(A+n) & \overset{\mathbb{P}\geq 1-\varepsilon}{=\!=\!=} & \Phi^{A+n} \\ {\scriptstyle \mathbb{P}=1}\Big| & & \Big|{\scriptstyle \therefore \mathbb{P}\geq 1-2\varepsilon} \\ F(A-n) & \overset{\mathbb{P}\geq 1-\varepsilon}{=\!=\!=} & \Phi^{A-n} \end{array}$$

- $=$ means equal
- $-$ means a Hamming distance of 1.

A research program

What other kinds of automorphisms can we rule out?

Example

Invertible functions $F : 2^\omega \to 2^\omega$ *that preserve a computably selected subsequence.*

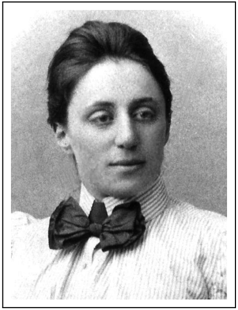
Example

Functions $F : 2^\omega \to 2^\omega$ *that map each set to a subset of itself.*

And so on.

Noether's theorem $\Rightarrow$ Rigidity of $\mathcal{D}_T$?

Each symmetry has a conserved quantity.
Analogously we could hope that each automorphism has a conserved quantity (the way those induced by permutations of ω do) and hence is trivial.



Emmy Noether

- Part IV: Aut($\mathcal{D}$) is $\mathcal{O}$-presentable

On a question of Schweber

In 2013, Noah Schweber asked

> *Is there any countable group G which we know can't be isomorphic to Aut($\mathcal{D}$)?*

Let p_i denote the ith prime number, and let $\oplus$ be the recursive join on ω. Let $\mathcal{O}$ be Kleene's Π^1_1-complete set and $\mathcal{O}'$ its Turing jump.

For any B, let G_B be the direct sum of $\mathbb{Z}/p_i\mathbb{Z}$ over all $i \in B \oplus \overline{B}$.

So G_B is a countably infinite abelian group.

Theorem

Aut($\mathcal{D}$) is not isomorphic to $G_{\mathcal{O}'}$.

I'll show this by showing that

Theorem

Aut($\mathcal{D}$) is $\Delta^0_1(\mathcal{O})$-presentable.

I.e., has a presentation which is recursive in $\mathcal{O}$, hence not $\geq_T B$. This will suffice because Richter, in her famous paper, showed that for all B, G_B has isomorphism type of degree $[B]_T$, i.e., all presentations of G_B have degree $\geq_T B$.

Note that if Aut($\mathcal{D}$) is finite then it is not isomorphic to G_B for any B, since the latter is countably infinite. So assume Aut($\mathcal{D}$) is infinite.

Slaman and Woodin showed that each automorphism π of $\mathcal{D}$ is represented by an arithmetic function in the sense that there is an n_0 such that for all π and all X, $\pi([X]_T) = [P(X)]_T$ where $P(X) = \{e\}(X^{(n_0)})$.

Let E be the set of those e for which P_e given by $X \mapsto \{e\}^{X^{(n_0)}}$ is an arithmetic representation of some automorphism.
We claim that the set E is Π^1_1: First, let F be the Π^1_1 set of all e for which

$$\forall A(P_e(A) \text{ is total}), \tag{1}$$

$$\forall A \forall B(A \leq_T B \to P_e(A) \leq_T P_e(B)), \text{ and} \tag{2}$$

$$\forall A \forall B(P(A) \equiv_T P(B) \to A \equiv_T B). \tag{3}$$

Then

$$E = \{e : e \in F \text{ and}$$

$$(\exists d \in F)\forall A(P_d(P_e(A)) \equiv_T A \text{ and } P_d(P_e(A)) \equiv_T A)\}.$$

The multiplication is given by defining $*$ by

$$P_{e_1 * e_2} = P_{e_1} \circ P_{e_2}$$

which is equivalent to

$$\forall A \forall B \forall C(B = P_{e_2}(A) \text{ and } C = P_{e_1}(B) \rightarrow C = P_{e_1 * e_2}(A))$$

We also have to mod out by equality of the automorphisms induced by e_1 and e_2, which we check by:

$$\forall A(P_{e_1}(A) \equiv_T P_{e_2}(A))$$

Overall, we get a subset of ω recursive in the Π^1_1-complete set Kleene's $\mathcal{O}$, with an $\mathcal{O}$-recursive group operation. This is then isomorphic to all of ω with an $\mathcal{O}$-recursive group operation, as desired.

Mahalo for your attention