Measuring Complexity of Maximal Matchings of Graphs

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Joint work with Stephen Flood, Matthew Jura, and Tyler Markkanen.
Definition
A matching in a graph $G = (V, E)$ is a subset $M \subseteq E$ for which each vertex is incident to at most one edge in $M$. A matching is perfect if every vertex is incident to exactly one edge in $M$. 
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Maximal matchings

Not every graph contains a perfect matching.
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Theorem (Steffens)
Every graph contains a maximal matching.
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Every graph contains a maximal matching.

Definition

- A matching $M$ is (weakly) maximal provided there is no matching $N$ with $M \subseteq N$. 
Maximal matchings

Not every graph contains a perfect matching.

**Theorem (Steffens)**

_Every graph contains a maximal matching._

**Definition**

- A matching $M$ is _weakly maximal_ provided there is no matching $N$ with $M \subset N$.
- A matching $M$ is _maximal_ provided there is no matching $N$ with $\text{supp}(M) \subset \text{supp}(N)$.

($\text{supp}(M)$ is the set of vertices incident to $M$.)
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How can you augment a matching?

Definition

An $M$-alternating path has edges alternating in and out of $M$.

An $M$-augmenting path is an $M$-alternating path that starts with an unmatched vertex and either ends in another unmatched vertex or is infinite.

If there is an $M$-augmenting path starting at $v$, then there is a matching $M'$ that improves the support of $M$ to include $v$ (and possibly one other vertex).
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Theorem (Steffens)

A graph has a perfect matching iff for any matching $M$ and unmatched $v$, there is an $M$-augmenting path starting at $v$. 
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Corollary

Every graph has a maximal matching.
Steffens’ Theorem

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Really, maximality seems to be a corollary to the proof of this theorem.
Our goal

Use Reverse Mathematics to understand the strength of both of these theorems.
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The plan:

1. Complete classification for locally finite graphs.
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The plan:

1. Complete classification for locally finite graphs.
2. Get a sense why the general case is much much much much harder to classify (probably).
Definition

- A graph is *locally finite* provided every vertex has finite degree.
- A graph is *bounded* provided there is a function $h : V \rightarrow \mathbb{N}$ s.t. $\forall x, y \in V (\{x, y\} \in E \rightarrow h(x) \geq y)$. 

Think: bounded = highly computable.
Definition

- A graph is *locally finite* provided every vertex has finite degree.
- A graph is *bounded* provided there is a function $h : V \to \mathbb{N}$ such that $\forall x, y \in V (\{x, y\} \in E \rightarrow h(x) \geq y)$.

Think: bounded = highly computable.
Theorem
The following are equivalent over RCA$_0$:

1. Every locally finite graph has a maximal matching.
2. A locally finite graph has a perfect matching iff it satisfies condition (A).
3. ACA$_0$. 
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Theorem
The following are equivalent over RCA₀:
1. Every bounded graph has a maximal matching.
2. A bounded graph has a perfect matching iff it satisfies condition (A).
3. WKL₀.
Proofs

Idea: Build a tree whose paths give perfect matchings.
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\[ \langle a_0, a_1, \ldots, a_n \rangle \in T \text{ iff } \{(0, a_0), (1, a_1), \ldots, (n, a_n)\} \text{ is a matching.} \]
Proofs

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Condition (A) guarantees the tree will be infinite.
Steffens for locally finite graphs implies ACA₀:
Reversals

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[Diagram of graph with dots and lines]

...
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Steffens for locally finite graphs implies ACA$_0$:

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Steffens for locally finite graphs implies $\text{ACA}_0$:

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\[
\begin{array}{c}
\text{---} \\
\text{---} \\
\text{---} \\
\text{---} \\
\text{---} \\
\text{---} \\
\end{array}
\]

Steffens for bounded graphs implies WKL₀:

\[
\begin{array}{c}
\text{---} \\
\text{---} \\
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\text{---} \\
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Graphs in general

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The problem: To use a larger matching, you must abandon a smaller matching.
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A subgraph is *independent* provided it has a perfect matching, and all perfect matchings are independent.
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**Lemma**
*Every graph has a maximal independent matching.*
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**Proof.**
Zorn’s lemma.
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Work in a countably coded $\beta_2$-model; build an increasing sequence of independent matchings; argue that the union is maximal.
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**Proof.**

Work in a countably coded $\beta_2$-model; build an increasing sequence of independent matchings; argue that the union is maximal.

Note: this is a proof in $\Pi^1_2$-CA.
Suppose $G$ satisfies condition (A)

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- If there is some $v$ not matched, take a maximal independent matching $M'$ of $G \setminus (V(M) \cup \{v\})$.

Note: we potentially need the maximal independent subgraph lemma infinitely often, but actually, exactly once.
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- If there is some $v$ not matched, take a maximal independent matching $M'$ of $G \setminus (V(M) \cup \{v\})$.
- Use an $M'$-augmenting path starting at $v$ to get a matching that includes $M$ and $v$, and whose complement satisfies condition (A).
A proof of Steffens’ theorem

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Take a maximal independent subgraph $H$ of $G$. 
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$G \setminus (H \cup N)$ satisfies condition (A), so by Steffens, has a perfect matching.
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Let $N$ be the set of vertices not in $H$ but adjacent only to vertices in $H$.

$G \setminus (H \cup N)$ satisfies condition (A), so by Steffens, has a perfect matching.

But the perfect matching would be independent in $G$, giving a larger independent matching. So any perfect matching of $H$ is a maximal matching of $G$. 
Lemma
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Proposition
Maximality implies $\Pi_1^1$-CA$_0$. Steffens implies $\Sigma_1^1$-AC$_0$. 
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Can we do better?

For any computable ordinal $\alpha$, there is a computable graph $G$ that satisfies condition (A), any perfect matching of which computes $0^{(\alpha)}$. 
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This would be enough to prove $\text{ATR}_0$, except we don’t know how to prove $G$ satisfies condition (A) without using $\Pi^1_1$-$\text{Tl}_0$. ($\Sigma^1_1$-$\text{DC}_0$)
The current picture

\[ \Pi^1_2 - \text{CA}_0^+ \rightarrow \text{Maximal Matching} \]

\[ \Pi^1_2 - \text{CA}_0 \rightarrow \text{Max Ind} \]

\[ \Pi^1_1 - \text{CA}_0 \leftarrow \text{Max Ind} \]

\[ \text{ATR}_0 \rightarrow +\Pi^1_1 - \text{TI}_0 \rightarrow \text{Steffens} \]

\[ \Sigma^1_1 - \text{AC}_0 \rightarrow \text{ACA}_0 \leftarrow \text{Locally Finite Steffens & Maximality} \]

\[ \text{ACA}_0 \leftarrow \text{Bounded Steffens & Maximality} \]

\[ \text{WKL}_0 \]
The End

Thanks!