Subclasses of the enumeration degrees arising from effective mathematics

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Prologue

- The *Turing degrees* measure the computability-theoretic complexity of elements of 2^ω (or ω^ω).
- We can code other mathematical objects as binary sequences and use the Turing degrees to measure their complexity.
- However, this does not always lead to a coherant measure of complexity; there may not be a "canonical" coding.
- The *enumeration degrees*, a natural extension of the Turing degrees, work in some circumstances where Turing degrees fail.
- E.g., the enumeration degrees can measure the complexity of continuous functions $f\colon [0,1] \to \mathbb{R}$. In fact, we get a proper subclass of the enumeration degrees: the *continuous degrees*.
- A larger subclass, the *cototal degrees*, arises naturally in symbolic dynamics and computable structure theory.

Part I: The Cototal Degrees

The Turing degrees are not always sufficient: Computable structure theory

Let $\mathcal{A}$ be a countable structure in a finite language L. A *presentation* of $\mathcal{A}$ is an ismorphic copy of $\mathcal{A}$ with universe ω.

Definition

The *degree spectrum* of a countable structure $\mathcal{A}$ is the collection $\mathrm{Spec}(\mathcal{A})$ of Turing degrees of presentations of $\mathcal{A}$.

When $\mathrm{Spec}(\mathcal{A})$ has a least element, we call it the *Turing degree* of $\mathcal{A}$.

Not all countable structure have a Turing degree.

Theorem (Richter 1977)

If a linear ordering $\mathcal{L}$ has Turing degree, then it is computable.

In this case, the enumeration degrees won't help much.

The Turing degrees are not always sufficient: Symbolic dynamics

The *shift operator* on 2^ω is the map taking an infinite binary sequence $\alpha \in 2^\omega$ to the unique $\beta \in 2^\omega$ such that $\alpha = a\beta$ for some $a \in \{0, 1\}$, i.e., the operator that erases the first bit of the sequence.

Definition

- A *subshift* is closed, shift-invariant subspace X of 2^ω.
- The *degree spectrum* of a subshift X is the set $\mathrm{Spec}(X)$ of Turing degrees of elements of the subshift.
- X is a *minimal subshift* if no nonempty $Y \subset X$ is a subshift.

If $\mathrm{Spec}(X)$ has a least element, then it could be considered as *Turing degree* of the subshift X.

Theorem (Hochman, Vanier 2017)

There is a minimal subshift X with no member of least Turing degree.

The enumeration degrees

Friedberg and Rogers introduced enumeration reducibility in 1959.

Informally: $A \subseteq \omega$ is *enumeration reducible* to $B \subseteq \omega$ ($A \leqslant_e B$) if there is a uniform way to enumerate A from an enumeration of B. (Selman proved that the uniformity condition can be dropped.)

Definition. $A \leqslant_e B$ if there is a c.e. set W such that

$$A = \{n \colon (\exists e)\, \langle n, e \rangle \in W \text{ and } D_e \subseteq B\},$$

where D_e is the eth finite set in a canonical enumeration.

The degree structure $\mathcal{D}_e$ induced by $\leqslant_e$ is called the *enumeration degrees*. It is an upper semi-lattice with a least element (the degree of all c.e. sets).

The total enumeration degrees

Proposition. $A \leqslant_T B$ iff $A \oplus \overline{A}$ is B-c.e. iff $A \oplus \overline{A} \leqslant_e B \oplus \overline{B}$.

This suggests a natural embedding of the Turing degrees into the enumeration degrees.

Proposition. The embedding $\iota \colon \mathcal{D}_T \to \mathcal{D}_e$, defined by

$$\iota(d_T(A)) = d_e(A \oplus \overline{A}),$$

preserves the order and the least upper bound (and even the jump).

Definition. A set A is *total* if $A \geqslant_e \overline{A}$ (or equivalently if $A \equiv_e A \oplus \overline{A}$). An enumeration degree is *total* if it contains a total set.

The image of the Turing degrees under the embedding ι is exactly the set of total enumeration degrees.

It is easy to see that there are nontotal enumeration degrees. In fact, the enumeration degrees are downwards dense (Gutteridge 1971).

Back to minimal subshifts

Given a minimal subshift X, we would like to characterize the set of Turing degrees of members of X.

Definition. The *language* of subshift $X \subseteq 2^\omega$ is the set

$$L_X = \{\sigma \in 2^{<\omega} : (\exists \alpha \in X)\ \sigma \text{ is a subword of } \alpha\}.$$

1. If X is minimal and $\sigma \in L_X$, then for every $\alpha \in X$, σ is a subword of α. So every element of X can enumerate the set L_X.
2. If we can enumerate L_X, then we can compute a member of X.

Theorem (Jeandel). A Turing degree $\mathbf{a}$ computes a member of the minimal subshift X if and only if $\mathbf{a}$ can enumerate L_X.

So the computability theoretic complexity of a minimal subshift X corresponds exactly to the enumeration degree of L_X.

The cotototal enumeration degrees

Jeandel noticed something special about L_X for a minimal subshift X.

- An enumeration of $\overline{L_X}$ allows us to eliminate branches that do not belong to X in a stage by stage manner.
- If w is word that appears along every branch that remains at stage s, then $w \in L_X$.
- The compactness of 2^ω ensures that we won't miss any word from the language using this process of enumeration.

So $L_X \leqslant_e \overline{L_X}$.

Definition. A set A is *cototal* if $A \leqslant_e \overline{A}$. An enumeration degree is *cototal* if it contains a cototal set.

Theorem (M., Soskova 2018). The cototal enumeration degrees are a dense substructure of the enumeration degrees.

Examples of cotototal enumeration degrees

Fact. Every total enumeration degree is cototal: $\overline{A \oplus \overline{A}} \equiv_e A \oplus \overline{A}$.

Definition (Carl von Jaenisch 1862)

Let $G = (V, E)$ be a graph. A set $M \subseteq V$ is *independent*, if no two members of M are edge related. M is *maximal* set, if every $v \in \overline{M}$ is edge related to a vertex in M.

Note. $\overline{M} \leqslant_e M$ because $v \in \overline{M}$ if and only if there is a $w \in M$ such that w and v are edge related.

Theorem (Andrews, Ganchev, Kuyper, Lempp, M., A. Soskova, and M. Soskova)

An enumeration degree is cototal if and only if it contains the complement of a maximal independent set for the graph $\omega^{<\omega}$.

Theorem (McCarthy). An enumeration degree is cototal if and only if it contains the complement of a maximal antichain in $\omega^{<\omega}$.

Cototal degrees and computable structure theory

Theorem (Montalbán)

A degree spectrum of a structure is not the Turing-upward closure of an F_σ set of reals in ω^ω, unless it is an *enumeration-cone* (the set of total/Turing degrees above some fixed enumeration degree).

In particular, it must be the cone above the enumeration degree of an e-pointed tree.

Definition (Montalbán). A tree $T \subseteq 2^{<\omega}$ is *e-pointed* if it has no dead ends and every infinite path $f \in [T]$ enumerates T.

Theorem (McCarthy). An enumeration degree is cototal if and only if it contains a (uniformly) e-pointed tree.

Corollary

A degree spectrum is the Turing-upward closure of an F_σ set of reals in ω^ω if and only if it is the enumeration-cone of a cototal degree.

Characterizing the degrees of minimal subshifts

Recall

- If X is a minimal subshift, then its spectrum (i.e., set of Turing degrees of members of X) is the enumeration-cone above the enumeration degree of L_X, the language of X.
- L_X is a cototal set (Jeandel).

Theorem (McCarthy)

Every cototal enumeration degree is the degree of the language of a minimal subshift.

So the cototal enumeration degrees arose independently in symbolic dynamics and in computable structure theory.

Part II: The Continuous Degrees

Example: every real number has a Turing degree

In computable analysis, coding is done via *names*.

Definition. $\lambda \colon \mathbb{Q}^+ \to \mathbb{Q}$ is a *name* of a real $x \in \mathbb{R}$ if for all rationals $\varepsilon > 0$ we have $|\lambda(\varepsilon) - x| < \varepsilon$.

Names can be easily coded as binary sequences, allowing us to transfer computability-theoretic notions to computable analysis. For example:

Definition. A function $f \colon \mathbb{R} \to \mathbb{R}$ is *computable* if there is a Turing functional that takes a name for any real $x \in \mathbb{R}$ to a name for $f(x)$.

- The binary expansion of a real x is computable from every name. (But this is nonuniform because of the dyadic rationals!)
- The binary expansion of x computes a name for x.
- This is the least Turing degree name for x; it is natural to take this as the *Turing degree* of x.

The Turing degrees are not always sufficient: Computable analysis

Definition. A *computable metric space* is a metric space $\mathcal{M}$ together with a countable dense sequence $Q^{\mathcal{M}} = \{q_n^{\mathcal{M}}\}_{n\in\omega}$ on which the metric is computable (as a function $\omega^2 \to \mathbb{R}$).

Example. The *Hilbert cube* is $[0,1]^\omega$ with the metric

$$d(\alpha,\beta) = \sum_{n\in\omega} |\alpha(n) - \beta(n)|/2^n.$$

Let $Q^{[0,1]^\omega}$ be the sequences of rationals in $[0,1]$ with finite support.

Other computable metric spaces include 2^ω, ω^ω, $\mathbb{R}$, and $\mathcal{C}[0,1]$.

Definition. $\lambda\colon \mathbb{Q}^+ \to \omega$ is a *name* of a point $x \in \mathcal{M}$ if for all rationals $\varepsilon > 0$ we have $d_{\mathcal{M}}(x, q^{\mathcal{M}}_{\lambda(\varepsilon)}) < \varepsilon$.

The Turing degrees are not always sufficient: Computable analysis

As before, the complexity of a point in a metric space can be captured through the collection of Turing degrees of names of this point.

Question (essentially Steffen Lempp). Do elements of computable metric spaces have least Turing degree names?

Useful fact. If $\beta \in [0,1]^\omega$ contains no dyadic rationals, then the sequence of binary expansions is computable from (every name for) β. But this sequence computes a name for β, which is therefore a least Turing degree name.

However, in general, least Turing degree names may not exist.

Theorem (M. 2004)

There is a $\beta \in [0,1]^\omega$ with no least Turing degree name.

The continuous degrees

Theorem (M. 2004)
There is a $\beta \in [0,1]^\omega$ with no least Turing degree name.

So how can we measure the complexity of points in a computable metric space?

Definition (M. 2004). If x and y are members of (possibly different) computable metric spaces, then $x \leqslant_r y$ if there is a uniform way to compute a name for x from every name for y.

This reducibility induces the *continuous degrees*.

Theorem (M. 2004). Every continuous degree contains a point from $[0,1]^\omega$ and a point from $C[0,1]$.

Embedding the continuous degrees into the e-degrees

For $\alpha \in [0,1]^\omega$, let

$$C_\alpha = \bigoplus_{i\in\omega} \{q \in \mathbb{Q} \colon q < \alpha(i)\} \oplus \{q \in \mathbb{Q} \colon q > \alpha(i)\}.$$

Observation. Enumerating C_α is exactly as hard as computing a name for α. So $\alpha \mapsto C_\alpha$ induces an embedding of the continuous degrees into the enumeration degrees.

- Elements of 2^ω, ω^ω, and $\mathbb{R}$ are mapped onto the *total* degree of their least Turing degree name (i.e., their Turing degree).
- It turns out that $x \in \mathcal{M}$ has nontotal (enumeration) degree iff it has no least Turing degree name.
- Every continuous degree is mapped to a cototal enumeration degree: $q < \alpha(i)$ iff there is some $p \leqslant \alpha(i)$ such that $q < p$.

So the continuous enumeration degrees extend the total degrees and form a subclass of the cototal degrees.

Nontotal continuous degrees: quick proof

Theorem (M. 2004). There is a nontotal continuous degree.

A quick proof was found independently by Kihara & Pauly and Mathieu Hoyrup.

Proof.

- If $x \in [0,1]^\omega$ has total degree, then there is a $y \in 2^\omega$ and Turing functionals Γ, Ψ that map (names of) x to (names of) y and back.
- The subspaces on which the functions induced by Γ and Ψ are inverses are homeomorphic (because computable functionals induce continuous functions).
- Subspaces of 2^ω are zero dimensional, so if $x \in [0,1]^\omega$ has total degree, then it is in one of *countably many* zero dimensional "patches".
- The Hilbert cube $[0,1]^\omega$ is strongly infinite dimensional, hence not a countable union of zero dimensional subspaces.
- So some $x \in [0,1]^\omega$ is not covered by one of these patches. □

Nontotal continuous degrees: neutral measures

The earliest known construction of an object of nontotal continuous degree was given by Leonid Levin in 1976.

Definition. $X \in 2^\omega$ is *ν-random* if there is a name λ of ν such that no ν-Martin-Löf test relative to λ covers X.

This definition is equivalent to ones of Levin 1976 and Reimann 2008.

Definition. ν is a *weakly neutral measure* if every $X \in 2^\omega$ is ν-random.

Levin constructed a *neutral measure*, which satisfies a slightly stronger condition, using Sperner's lemma, a combinatorial analogue of the Brouwer fixed point theorem.

Proposition (Day and M. 2013). If ν has Turing degree, then it is not weakly neutral.

So we have another proof that nontotal continuous degrees exist.

Nontotal continuous degrees: my proof

Theorem (M. 2004). There is a nontotal continuous degree.

My proof *also* relies on a nontrivial fact from topology, a generalization of Brouwer's fixed point theorem to multivalued functions on an infinite dimensional space.

Theorem (Eilenberg and Montgomery 1946). Assume that $\Psi\colon [0,1]^\omega \to [0,1]^\omega$ is a multivalued function with closed graph such that $\Psi(\alpha)$ is nonempty and convex for each $\alpha \in [0,1]^\omega$. Then Ψ has a fixed point α (i.e., $\alpha \in \Psi(\alpha)$).

I constructed such a Ψ so that the fixed points have nontotal continuous degree, proving the theorem.

This approach gives more information because Ψ is effective enough that (the names for) its fixed points form a Π^0_1 class.

Prop. Every PA total degree bounds a nontotal continuous degree.

Intervals containing nontotal continuous degrees

Prop. Every PA total degree bounds a nontotal continuous degree.

The reverse is also true:

Prop. Every nontotal continuous degree bounds a PA total degree.

Aside. The proof invokes topology again, this time using a constructive counterexample of V. P. Orevkov: he gave a continuous retraction of (the constructive points of) the unit square $[0,1]^2$ onto its boundary $\partial([0,1]^2)$.

So a total degree $\mathbf{a}$ is PA if and only if it bounds a nontotal continuous degree. Relativizing this fact we obtain:

Theorem (M. 2004). Let $\mathbf{b} \leqslant \mathbf{a}$ be total. There is a nontotal continuous degree $\mathbf{c} \in (\mathbf{b}, \mathbf{a})$ if and only if $\mathbf{a}$ is PA relative to $\mathbf{b}$.

Also note that continuous degrees are not downwards dense, hence they differ from both the enumeration degrees and the cototal degrees.

Characterizing the continuous enumeration degrees

As it turns out, the contiunous enumeration degrees have a very simple characterization inside the enumeration degrees.

Definition

An enumeration degree $\mathbf{a}$ is *almost total* if whenever $\mathbf{b} \nleqslant \mathbf{a}$ is total, $\mathbf{a} \vee \mathbf{b}$ is also total.

In other words, an enumeration degree is almost total if adding any new total information takes it to a total degree.

Note. The join of any two total degrees is total, so total degrees are almost total.

Are there nontotal almost total degrees?

Continuous degrees are almost total

Are there nontotal almost total degrees? Yes!

Recall. If $\beta \in [0,1]^\omega$ contains no dyadic rationals, then β is equivalent the join of the binary expansions of its coordinates, which has total degree.

Fact (Cai, Lempp, M., Soskova 2014 (unpublished)). Continuous enumeration degrees are almost total.

Proof. Take $\alpha \in [0,1]^\omega$ and $x \in [0,1]$ such that $x \not\leqslant_r \alpha$. Define $\beta \in [0,1]^\omega$ by $\beta(n) = (\alpha(n) + x)/2$. Note that

- No component of β is rational, so β has total degree.
- $\alpha \oplus x \equiv_r \beta \oplus x$, hence it is also total. □

There are nontotal continuous degrees, so there are nontotal almost total degrees. This is the only way we know how to produce nontotal almost total degrees. (In particular, we have no "direct" construction.)

Almost total degrees are continuous

Theorem (Andrews, Igusa, M., Soskova). Almost total degrees are continuous.

We used a series of implications:

$$\begin{aligned} \text{Almost total} &\implies \text{Uniformly codable} \\ &\implies \text{Contains a holistic set} \\ &\implies \text{Continuous.} \end{aligned}$$

- All known constructions of nontotal continuous degrees involve a nontrivial topological component.
  - Conversely, the fact that the Hilbert cube is not a countable union of subspaces of Cantor space follows easily from the fact that there is a nontotal continuous degrees in every cone.

So a purely topological fact is reflected in the structure of the enumeration degrees.

Uniform Codability and Holistic sets

Definition. Let $A \subseteq \omega$. Call $U \subseteq 2^\omega$ a *$\Sigma^0_1\langle A\rangle$ class* if there is a set of strings $W \leqslant_e A$, such that

$$U = [W] = \{X \in 2^\omega : (\exists \sigma \in W)\ X \succeq \sigma\}.$$

A *$\Pi^0_1\langle A\rangle$ class* is the complement of a $\Sigma^0_1\langle A\rangle$ class.

Note that a $\Pi^0_1\langle A \oplus \overline{A}\rangle$ class is just a $\Pi^0_1[A]$ class in the usual sense.

Definition. $A \subseteq \omega$ is *codable* if there is a nonempty $\Pi^0_1\langle A\rangle$ class P such that every $X \in P$ enumerates A. If there is a c.e. operator W such that $A = W^X$ for every $X \in P$, then A is *uniformly codable*.

Definition. $S \subseteq \omega^{<\omega}$ is *holistic* if for every $\sigma \in \omega^{<\omega}$,

1. $(\forall n)\ \sigma^\frown(2n)$ and $\sigma^\frown(2n+1)$ are not both in S,
2. If $\sigma \in S$, then $(\exists n)\ \sigma^\frown(2n+1) \in S$.
3. If $\sigma \notin S$, then $(\forall n)\ \sigma^\frown(2n) \in S$,

Definability in the enumeration degrees

Theorem (Cai, Ganchev, Lempp, M., and Soskova 2016). The total degrees are first order definable in the enumeration degrees (as a partial order).

The definition is "natural". It builds on work of Kalimullin (2003) and Ganchev and Soskova (2015).

Corollary (AIMS). The continuous degrees are definable in the enumeration degrees.

Recall that if $\mathbf{a}$ and $\mathbf{b}$ are total degrees, then $\mathbf{a}$ is PA above $\mathbf{b}$ iff there is a nontotal continuous degree $\mathbf{c} \in (\mathbf{b}, \mathbf{a})$.

Corollary (AIMS). The relation "$\mathbf{a}$ is PA above $\mathbf{b}$" (on total degrees) is first order definable in the enumeration degrees.

It is not known to be definable in the Turing degrees.

To finish: a unifying example

Kihara and Pauly assigned (enumeration) degrees to point in any second countable T_0 topological space.

Definition (Kihara and Pauly)

If $\mathcal{X}$ has countable basis $\{N_i\}_{i\in\omega}$, then the degree of $x \in \mathcal{X}$ to the enumeration degree of $\{i : x \in N_i\}$.

The point degree spectrum of a topological space $\mathcal{X}$ (i.e., the degrees of points $\mathcal{X}$) is a subclass of the enumeration degrees.

1. $\mathrm{Spec}(2^\omega) = \mathrm{Spec}(\omega^\omega) = \mathrm{Spec}(\mathbb{R}) = \mathcal{D}_T$;
2. $\mathrm{Spec}([0,1]^\omega) = \mathrm{Spec}(C[0,1]) = \mathcal{D}_r$;
3. $\mathrm{Spec}(S^\omega) = \mathcal{D}_e$, where $S = \{\emptyset, \{0\}, \{0,1\}\}$ is the Sierpinski space.

Theorem (Kihara). If $\mathcal{X}$ is a sufficiently effective, second countable G_δ spaces (i.e., every closed set is G_δ), then every point in $\mathcal{X}$ has cototal degree. Conversly, all cototal degrees arise in this way.

Thank you!