Generically Computable Structures

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Outline

New notion of *generically computable structures*

Faithful generically computable equivalence structures

Characterization of equivalence structures with faithful generically computable copies

Coarsely computable structures

Generically computable isomorphisms and categoricity

Generic character

Joint work with Wesley Calvert and Valentina Harizanov

Generically computable sets

Jockusch and Schupp introduced these notions (J. London Math. Soc. 2012)

Definition

Let $S \subseteq \omega$.

1. *S is generically computable if there is a partial computable function $\Phi : \omega \to 2$ such that $\Phi = \chi_S$ on the domain of Φ, and such that the domain of Φ has asymptotic density 1.*
2. *S is coarsely computable if there is a computable set T such that $S \triangle T$ has asymptotic density 0.*

There is a coarsely computable c.e. set which is not generically computable and a generically computable c.e. set which is not coarsely computable.

Density in $\omega \times \omega$

For $A \subseteq \omega$, $\delta(A) = lim_{n \to \infty} |A \cap n| / n$

Upper density is $limsup_{n \to \infty} |A \cap n| / n$

For $C \subseteq \omega \times \omega$, let $\delta(C) = lim_{n \to \infty} |C \cap (n \times n| / n^2$

Lemma

For $A \subseteq \omega$, $\delta(A) = \delta \iff \delta(A \times A) = \delta^2$.

On the other hand, we have:

Theorem

There is a computable dense $C \subset \omega \times \omega$ such that for any infinite c.e. set $A \subset \omega$, $A \times A$ is not a subset of C.

Generically Computable Relations

Definition

Let R be a relation on ω.

1. *R is generically computable if there is a partial computable $\phi : \omega \times \omega \to 2$ with $\Phi = \chi_R$ on the domain of ϕ, and a dense c.e. set A with $A \times A \subseteq Dom(\phi)$.*
2. *An equivalence relation R is generically computable if there is a p.c. $\phi : \omega \times \omega \to 2$ such that $\phi = \chi_R$ on $Dom(\phi)$, and a dense c.e. set A with $A \times A \subseteq Dom(\phi)$.*
3. *An equivalence relation R is generically c.e. if there is a c.e. equivalence relation S on a c.e. set B and a dense c.e. set $A \subseteq B$ such that R agrees with S on $A \times A$.*
4. *An equiv. relation R is strongly generically computable if there is a computable equiv. relation S on ω and a dense c.e. set A such that R agrees with S on $A \times A$.*

Faithful Relations

Definition

If R is a relation on ω and A is a subset of ω, we say that A is R-faithful if whenever $a \in A$ and either $R(a, b)$ or $R(b, a)$, then $b \in A$.

A is R-faithful for an equivalence relation R IFF for any R-equivalence class C, either $C \subseteq A$ or $C \cap A = \emptyset$.

R is *faithfully generically computable* if the set A in the previous definition is R-faithful and similarly for the other notions.

Computable Equivalence Structures

Definition

For any equivalence relation E on ω, the character $\chi(E)$ is $\{(k, n) : E$ has at least n equivalence classes of size $k\}$.

Character K is *unbounded* if $\{k : (k, 1) \in K\}$ is unbounded.

Lemma

For any c.e. equivalence relation R on a c.e. set A, the character $\chi(R)$ is a Σ^0_2 set.

Proposition

For any Σ^0_2 character K

1. *There is a computable equivalence structure with character K and infinitely many infinite classes.*
2. *There is a c.e. structure with character K and one infinite class (or k for any finite k).*

s_1-Functions

Definition

The function $f : \omega^2 \to \omega$ is said to be an s_1-function if the following hold:

1. *For every i and s, $f(i,s) \leq f(i,s+1)$.*
2. *For every i, the limit $m_i = lim_s f(i,s)$ exists.*
3. *For every i, $m_i < m_{i+1}$.*

Lemma

Let $\mathcal{A} = (\omega, E)$ be a c.e. equivalence structure with no infinite equivalence classes and an unbounded character. Then there is a computable s_1-function f such that $\mathcal{A}$ contains an equivalence class of size m_i for all i, where $m_i = lim_s f(i,s)$.

More about s_1-Functions

Lemma

For any Σ_2^0 character K which possesses a computable s_1-function, there is a computable equivalence structure $\mathcal{E}$ with character K and no infinite equivalence classes.

Generically Computable Equivalence Relations

Theorem

Let $\mathcal{E} = (\omega, E)$ be an equivalence relation such that either

1. *$\mathcal{E}$ has an infinite equivalence class, or*
2. *there is a finite k such that $\mathcal{E}$ has infinitely many classes of size k, or*
3. *$\chi(\mathcal{E})$ has an infinite Σ^0_2 subset with an s_1-function.*

Then $\mathcal{E}$ has a faithfully strongly generically computable copy.

Proof

Let B be an infinite class in $\mathcal{E}$.

Let $\mathcal{C}$ have an infinite class A which is a computable dense set, and with $(\omega \setminus B, E)$ isomorphic to $(\omega \setminus A, R)$.

So (ω, R) is a copy of $\mathcal{E}$ and A is R-faithful.

Let (ω, S) consist of two infinite classes A and $\omega \setminus A$.

Then S is computable and agrees with R on A.

For the second case, take B to consist of infinitely many classes of size k.

For the third case, take B to have a Σ_2^0 character with an s_1-function.

Unfaithful Structures

Theorem

Every equivalence structure $\mathcal{E} = (\omega, E)$ has a strongly generically computable copy.

Proof Sketch: This follows from previous Theorem unless the character is unbounded.
Let $C = \{k : (k, 1) \in \chi(\mathcal{E})\}$.
Then choose a class B_k of size k for each k.
Let B consist of one element from each B_k.
Let A be a dense computable set and let R be equality on A.
Define R on $\omega \setminus A$ so that (ω, R) is isomorphic to $\mathcal{E}$.
Define S to be equality.

Then (ω, S) is computable and agrees with (ω, R) on A.

Faithful Structures

Theorem

Let $\mathcal{E} = (\omega, E)$ be an equivalence structure. Then the following are equivalent:

(a) *$\mathcal{E}$ has a faithfully strongly generically computable copy*

(b) *$\mathcal{E}$ has a faithfully generically computable copy*

(c) *$\mathcal{E}$ has a faithful generically c.e. copy*

(d) *$\mathcal{E}$ has an infinite faithful substructure with a computable copy*

(e) *$\mathcal{E}$ has an infinite faithful substructure with a c.e. copy*

(f) *either (i) $\mathcal{E}$ has an infinite equivalence class, or (ii) there is a finite k such that $\mathcal{E}$ has infinitely many classes of size k, or (iii) $\chi(\mathcal{E})$ has an infinite Σ^0_2 subset with an s_1-function.*

Coarsely Computable Structures

Definition

An equivalence structure $\mathcal{E} = (\omega, E)$ is coarsely computable if there is a computable equivalence relation R and a set A of density one such that for $a, b \in A$, $aRb \iff aEb$. If (A, E) is a faithful substructure of $\mathcal{E}$, then $\mathcal{E}$ is faithfully coarsely computable.

Strongly generically computable implies coarsely computable, so previous results apply.

We will construct a faithfully coarsely computable structure with no faithfully generically computable copy.

Examples

Example: Let (ω, E) be the canonical structure with one classe of every finite size k
The equivalence classes of (ω, E) are $\{0\}, \{1,2\}, \{3,4,5\}, \ldots$
The first k classes have $1+2+\cdots+k = k(k+1)/2$ elements.
For any set K, A_K is the classes of size k for $k \in K$, under E.

Lemma

If K is a dense set, then A_K is also a dense set.

Lemma

For any dense co-infinite K, there is a faithfully coarsely computable structure with character $\{(k,i) : k \in K, i \leq 2\}$.

Proof: Let (ω, E) be as above. Define R to be E on A_k and to partition $\omega \setminus K$ into one class of size k for each $k \in K$.

Coarsely but not Generically

Observation

There is a dense set K with no infinite Σ_2^0 subset.

For such K we get a faithfully coarsely computable structure with no faithfully generically computable copy.

That is, the character $\{(k, i) : k \in K, i \leq 2\}$ can have no infinite Σ_2^0 subset since K has no infinite Σ_2^0 subset.

Generically Computable Functions

Definition

A function F is generically computable if there is a p.c. function ϕ such that $\phi = F$ on $Dom(\phi)$, and $Dom(\phi)$ is dense.

Definition

$F : \mathcal{A} \to \mathcal{B}$ is a generically computable isomorphism if there is a p.c. θ such that $F = \theta$ on $Dom(\phi)$, and $Dom(\phi)$ and $Rng(\phi)$ are both dense.

Proposition

Structures $\mathcal{A}$ and $\mathcal{B}$ are generically computably isomorphic if and only if there is an isomorphism $F : \mathcal{A} \to \mathcal{B}$ such that both F and F^{-1} are generically computable.

Coarsely Computable Functions

Definition

A function F is coarsely computable if there is a total computable ϕ such that $\{n : F(n) = \phi(n)\}$ is dense.

Definition

1. *An isomorphism $F : \mathcal{A} \to \mathcal{B}$ is coarsely computable if there is a total computable θ such that $C = \{x : \theta(x) = F(x)\}$ and $F[C]$ are both dense.*
2. *A set isomorphism $F : \mathcal{A} \to \mathcal{B}$ is a weakly coarsely computable isomorphism if there is a total computable θ which is an isomorphism of the substructure $C = \{x : \theta(x) = F(x)\}$ to $F[C]$ and both sets are dense.*

Generically Computably Isomorphic Structures

An equivalence structure $\mathcal{A}$ is a *(1,2)-structure* if it consists of infinitely many classes of size 1 and of size 2.

Fact: There are computable (1,2) structures which are not computably isomorphic.

Theorem

If $\mathcal{A}$ and $\mathcal{B}$ are computable (1,2)-structures and the classes of size 2 are dense in each, then $\mathcal{A}$ and $\mathcal{B}$ are generically computably isomorphic.

Sketch: The classes of size 2 form a c.e. set in each structure.

Let $\{a_0, a_1\}, \{a_2, a_3\}, \dots$ enumerate the classes of size 2 in $\mathcal{A}$ and $\{b_0, b_1\}, \{b_2, b_3\}, \dots$ enumerate the classes of size 2 in $\mathcal{B}$.

Define $\phi(a_n) = b_n$ for each n. Now extend this arbitrarily to the classes of size 1 to obtain F.

On the Other Hand

Theorem

There are computable (1,2)-structure $\mathcal{A}$ and $\mathcal{B}$ such that the classes of size 1 are dense in each, but $\mathcal{A}$ and $\mathcal{B}$ are notgenerically computably isomorphic.

Sketch: Just let the classes of size 2 compose a simple c.e. set of density 0.

Coarsely Computably Isomorphic Structures

Theorem

If $\mathcal{A} = (\omega, E_A)$ and $\mathcal{B} = (\omega, E_B)$ be are isomorphic (1,2)-structures such that the classes of size 1 are dense in both, then $\mathcal{A}$ and $\mathcal{B}$ are coarsely computably isomorphic.

Sketch: Let U_A be the classes of size 1 in A and U_B the classes of size 1 in $\mathcal{B}$. Let $U = U_A \cap U_B$.

Then the identity is an isomorphism of (U, E_A) to (U, E_B).

If $U_A \setminus U$ and $U_B \setminus U$ have different cardinalities, just remove a subset of density 0 from the larger set.

A More General Result

Theorem

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are computable $(1, 2)$-structures such that the asymptotic density of the classes of size 2 in each both equal a computable real q. Then $\mathcal{A}$ and $\mathcal{B}$ are weakly coarsely computably isomorphic.

Idea of Proof: Define the map in stages as we see the proportion of classes of size 2 approach q so the amount of "errors" tends to 0.

Current and Future Work

Other notions of almost computability

Other equivalence structures beyond (1,2)-structures

Other structures, such as orderings, Boolean algebras, injection structures, p-groups

THANK YOU