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# Automorphism groups of substructure lattices

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Joint work with Rumen Dimitrov and Andrei Morozov.

- Let $\mathcal{D}$ be the set of all Turing degrees.

  Let $I \neq \emptyset$ and $I \subseteq \mathcal{D}$. $I$ is called a *Turing ideal* if:

  (1) $(\forall \mathbf{b} \in I)(\forall \mathbf{a})[\mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{a} \in I]$

  (2) $(\forall \mathbf{a} \in I)(\forall \mathbf{b} \in I)[\mathbf{a} \vee \mathbf{b} \in I]$

- Let $I$ be a Turing ideal. Let $\mathcal{M}$ be a computable structure.

  (1) $Aut_I(\mathcal{M})$ is the set of all $\mathbf{d}$-computable automorphisms of $\mathcal{M}$ for any $\mathbf{d} \in I$.

  (2) If $I = \{\mathbf{s} : \mathbf{s} \leq \mathbf{d}\}$, then $Aut_I(\mathcal{M})$ is also denoted by $Aut_{\mathbf{d}}(\mathcal{M})$.

- If $\mathcal{M} = (\omega, =)$, then $Aut(\mathcal{M})$=$Sym(\omega)$, the symmetric group of $\omega$.

  $Sym_{\mathbf{d}}(\omega) = \{f \in Sym(\omega) : \deg(f) \leq \mathbf{d}\}$

## Lattices of vector spaces and their automorphisms

- $V_\infty$: computable $\aleph_0$-dimensional vector space over a computable field $F$ with uniformly computable dependence relations $(D_n)_{n\in\omega}$ (dependence algorithm)

- Can think of the elements of $V_\infty$, the vectors, as infinite sequences of elements of $F$ with only finitely many nonzero components.

- Pointwise vector addition and scalar multiplication.

- $V_\infty$ has a standard computable basis.

- Let $\mathcal{L}$ denote the lattice of all subspaces of $V_\infty$: $(\mathcal{L}, \subseteq, \cap, +)$

  $U + W = cl(U \cup W)$

- Sublattice $\mathcal{L}_{\mathbf{d}}(V_\infty)$ =$\{V \in \mathcal{L} : V$ is $\mathbf{d}$-computably enumerable$\}$

- A (sub)space $V \subseteq V_\infty$ is c.e. if $V$ is a c.e. set.

- $\mathcal{L}_{\mathbf{0}}(V_\infty) = \mathcal{L}(V_\infty)$, the lattice of c.e. subspaces of $V_\infty$

  Modular lattice: $x \leq b \Rightarrow [x \vee (a \wedge b) = (x \vee a) \wedge b]$

- (Guichard) The automorphisms of $\mathcal{L}(V_\infty)$ are induced by $\mathbf{1-1}$ and onto *computable* semilinear transformations.
  Hence there are countably many automorphisms of $\mathcal{L}(V_\infty)$.

- $\langle \mu, \sigma \rangle$ is a *semilinear* transformation if $\mu : V_\infty \to V_\infty$, $\sigma$ is an automorphism of $F$, and for every $u, v \in V_\infty$ and $a, b \in F$:

$$\mu(au + bv) = \sigma(a)\mu(u) + \sigma(b)\mu(v)$$

- **Conjecture** (Ash) The automorphisms of $\mathcal{L}^*(V_\infty)$ are induced by semilinear transformations with finite dimensional kernels and co-finite dimensional images in $V_\infty$.

- For c.e. sets, both lattices $\mathcal{E}$ and $\mathcal{E}^*$ have $2^{\aleph_0}$ automorphisms.

- By $GSL_{\mathbf{d}}$ we denote the group of $1-1$ and onto semilinear transformations $\langle \mu, \sigma \rangle$ such that $deg(\mu) \leq \mathbf{d}$ and $deg(\sigma) \leq \mathbf{d}$.

- (i) Every $\Phi \in Aut(\mathcal{L}_{\mathbf{d}}(V_\infty))$ is induced by some $\langle \mu, \sigma \rangle \in GSL_{\mathbf{d}}$.

  (ii) If $\Phi \in Aut(\mathcal{L}_{\mathbf{d}}(V_\infty))$ is induced by $\langle \mu, \sigma \rangle \in GSL_{\mathbf{d}}$ and by some other $\langle \mu_1, \sigma_1 \rangle \in GSL_{\mathbf{d}}$, then there is $\gamma \in F$ such that

$$(\forall v \in V_\infty)[\mu(v) = \gamma \mu_1(v)]$$

- **Proof**. Automorphism $\Phi$ of $\mathcal{L}_{\mathbf{d}}(V_\infty)$ determines a unique automorphism $\Psi$ of $\mathcal{L}$. By the fundamental theorem of projective geometry applied to the lattice $\mathcal{L}$, $\Psi$ is induced by a semilinear transformation $\langle \mu, \sigma \rangle$. Note that $\langle \mu, \sigma \rangle$ also generates $\Phi$. Will show that $deg(\mu) \leq \mathbf{d}$ and $deg(\sigma) \leq \mathbf{d}$.

- Let $\alpha_0, \alpha_1, \alpha_2, ...$ be a fixed computable enumeration of the elements of the field $F$. Let $v_0, v_1, v_2, ...$ be a computable enumeration of a computable basis of $V_\infty$.

- Define the following computable subspaces of $V_\infty$:

  $V_1 = cl(\{v_0, v_2, v_4, ...\})$

  $V_2 = cl(\{v_1, v_3, v_5, ...\})$

  $V_3 = cl(\{v_0 + v_1, v_2 + v_3, v_4 + v_5, ...\})$

  $V_4 = cl(\{v_1 + v_2, v_3 + v_4, v_5 + v_6, ...\})$

  $V_5 = cl(\{v_0 + \alpha_0 v_1, v_2 + \alpha_1 v_3, v_4 + \alpha_2 v_5, ...\})$

- Suppose that $\Phi(V_i) = W_i$ for $i = 1, \ldots, 5$, and note that $W_i \in \mathcal{L}_{\mathbf{d}}(V_\infty)$.

- To prove that $deg(\mu) \leq \mathbf{d}$, suppose that $\mu(v_0) = w_0$ for some fixed $w_0$. Inductively suppose that $\mu(v_{2i}) = w_{2i}$ has been found $\mathbf{d}$-computably. To find $\mathbf{d}$-computably $\mu(v_{2i+1})$, we let $w_{2i+1}$ be the least $y \in W_2$ such that $w_{2i} + y \in W_3$. Then we have $\mu(v_{2i+1}) = w_{2i+1}$. Next, to find $\mathbf{d}$-computably $\mu(v_{2i+2})$, we let $w_{2i+2}$ be the least $y \in W_1$ such that $w_{2i+1} + y \in W_4$. Then we have $\mu(v_{2i+2}) = w_{2i+2}$.

- Finally, to find $\mathbf{d}$-computably $\sigma(\alpha_i)$, we look for the least $w \in W_5$ and $\beta \in F$ such that $w = w_{2i} + \beta w_{2i+1}$. Then we have $\sigma(\alpha_i) = \beta$.

- If our choice for $\mu(v_0)$ is a scalar multiple of the original $w_0$, that is, $\mu(v_0) = \gamma w_0$, then $\mu(v_i) = \gamma w_i$ for $i \geq 1$.

- Every automorphism of $\mathcal{L}_I(V_\infty)$ is defined on the one-dimensional subspaces of $V_\infty$ and can be uniquely extended to an automorphism of the entire lattice $\mathcal{L}$.

  Hence we can identify the automorphisms of $\mathcal{L}_I(V_\infty)$ with their extensions to automorphism of $\mathcal{L}$.

- Every automorphism of $\mathcal{L}$ is induced by some $\mathbf{d}$-computable semilinear transformation and its restriction to $\mathcal{L}_{\mathbf{d}}(V_\infty)$ is an automorphism of $\mathcal{L}_{\mathbf{d}}(V_\infty)$.

- $Aut(\mathcal{L}_I(V_\infty)) = \bigcup_{\mathbf{d}\in I} Aut(\mathcal{L}_{\mathbf{d}}(V_\infty))$

- $Aut(\mathcal{L}) = \bigcup_{\mathbf{d}\in \mathcal{D}} Aut(\mathcal{L}_{\mathbf{d}}(V_\infty))$

- (Dimitrov, Harizanov and Morozov)

  For any pair of Turing ideals $I, J$ we have

  $Aut(\mathcal{L}_I(V_\infty)) \hookrightarrow Aut(\mathcal{L}_J(V_\infty)) \Leftrightarrow I \subseteq J$

- **Corollary**. For any pair $\mathbf{a}, \mathbf{b}$ of Turing degrees we have

$$(Aut(\mathcal{L}_{\mathbf{a}}(V_\infty)) \hookrightarrow Aut(\mathcal{L}_{\mathbf{b}}(V_\infty))) \Leftrightarrow \mathbf{a} \leq \mathbf{b}$$

- (Morozov)

$$(Sym_{\mathbf{a}}(\omega) \hookrightarrow Sym_{\mathbf{b}}(\omega)) \Leftrightarrow (\mathbf{a} \leq \mathbf{b})$$

## Turing degree spectrum of $GSL_{\mathbf{d}}$

- *Turing degree spectrum* of a structure $\mathcal{M}$:

$$DgSp(\mathcal{M}) = \{deg(\mathcal{A}) : \mathcal{A} \cong \mathcal{M}\}$$

- (Knight) Turing degree spectrum of a structure is either a singleton or is closed upward in the set $\mathcal{D}$ of all Turing degrees.

- (Dimitrov, Harizanov and Morozov)

$$DgSp(GSL_{\mathbf{d}}) = \{\mathbf{c} \in \mathcal{D} : \mathbf{c} \geq \mathbf{d}''\}$$

- (Jockusch and Richter) The *degree of the isomorphism type* of a structure $\mathcal{M}$, if it exists, is the least Turing degree in the Turing degree spectrum of $\mathcal{M}$.

- A permutation is $1_{\mathsf{inf}}2_{\mathsf{inf}}$ if it is a product of infinitely many 1-cycles and infinitely many 2-cycles.

  A permutation is $1_{\mathsf{inf}}2_{\mathsf{fin}}$ if it is a product of infinitely many 1-cycles and finitely many 2-cycles.

- Let $G$ be an $X$-computable group, and let $H : Sym_{\mathbf{0}}(\omega) \hookrightarrow G$ be an arbitrary embedding. Suppose that for every $1_{\mathsf{inf}}2_{\mathsf{inf}}$ permutation $p \in Sym_{\mathbf{0}}(\omega)$, the image $H(p)$ is not a conjugate of the image of any $1_{\mathsf{inf}}2_{\mathsf{fin}}$ permutation in $Sym_{\mathbf{0}}(\omega)$.

  Then $\mathbf{0}'' \leq \mathsf{deg}(X)$.

- Let $A$ be a $\Pi^0_2$-complete set and let $R(n,t)$ be a computable predicate such that

$$n \in A \Leftrightarrow (\exists^\infty t)\, R(n,t).$$

  We can prove that $A \leq_T X$.

- Can construct an embedding $H : Sym_{\mathbf{0}}(\omega) \hookrightarrow GSL_{\mathbf{0}}$ such that the images of any $1_{\mathsf{inf}}2_{\mathsf{inf}}$ and $1_{\mathsf{inf}}2_{\mathsf{fin}}$ permutations from $Sym_{\mathbf{0}}(\omega)$ cannot be conjugates in $GSL_{\mathbf{0}}$. By the previous result, conclude that $\mathbf{0}''$ is computable in any copy of $GSL_{\mathbf{0}}$.

  We can construct a copy of $GSL_{\mathbf{0}}$, which is computable in $\mathbf{0}''$.

- This can be easily relativized to any Turing degree $\mathbf{d}$.

## Lattices of Boolean algebras and their automorphisms

- Let $\mathcal{B}_\eta$ be the interval Boolean algebra over the order type $\eta$.

  We identify the elements of $\mathcal{B}_\eta$ with finite unions and intersections of left-closed right-open intervals of the countable dense linear order $\eta$ of the rationals.

- Let $\mathcal{L}_I(\mathcal{B}_\eta)$ be the lattice of all subalgebras of $\mathcal{B}_\eta$, which are c.e. in some $\mathbf{d} \in I$.
  $\mathcal{L}_\mathbf{d}(\mathcal{B}_\eta)$ is the lattice of all subalgebras of $\mathcal{B}_\eta$, which are c.e. in $\mathbf{d}$.
  $\mathcal{L}_\mathbf{0}(\mathcal{B}_\eta)$ is also denoted by $\mathcal{L}(\mathcal{B}_\eta)$.

- Every automorphism of $Aut(\mathcal{L}_\mathbf{d}(\mathcal{B}_\eta))$ is induced by a $\mathbf{d}$-computable automorphism of $\mathcal{B}_\eta$ and every $\mathbf{\Phi} \in Aut(\mathcal{L}_I(\mathcal{B}_\eta))$ is induced by a $\mathbf{d}$-computable automorphism of $\mathcal{B}_\eta$ for some $\mathbf{d} \in I$.

(Dimitrov, Harizanov and Morozov)

- (i) $Aut(\mathcal{L}_I(\mathcal{B}_\eta)) \cong Aut_I(\mathcal{B}_\eta)$

  (ii) $Aut_I(\mathcal{B}_\eta) \hookrightarrow Aut_J(\mathcal{B}_\eta) \Leftrightarrow I \subseteq J$

  (iii) $Aut(\mathcal{L}_I(\mathcal{B}_\eta)) \hookrightarrow Aut(\mathcal{L}_J(\mathcal{B}_\eta)) \Leftrightarrow I \subseteq J$

- The degree of the isomorphism type of the group $Aut_{\mathbf{d}}(B_\eta)$ is $\mathbf{d}''$.

# THANK YOU!