Completions of PA and ω-models of KP

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Muchnik reducibiility

Muchnik reducibility lets us compare "problems", where a *problem* is a subset of either Cantor space 2^ω or Baire space ω^ω.

Definition. *P is Muchnik reducible to Q*, or $P \leq_w Q$, if every $f \in Q$ computes some $g \in P$.

We are interested in problems of the form $[T]$, where T is a computable subtree of $2^{<\omega}$ or $\omega^{<\omega}$.

For simplicity, we write $T_1 \leq_w T_2$ if $[T_1] \leq_w [T_2]$.

T_{PA}

Fact. We write T_{PA} for the usual computable subtree of $2^{<\omega}$ whose paths represent the completions of PA.

Scott. For all computable trees $T \subseteq 2^{<\omega}$, $T \leq_w T_{PA}$.

Thus, among binary branching trees, T_{PA} lies on top under Muchnik reducibility.

KP

Kripke-Platek set theory is a weak version of set theory. There are the usual axioms of extent, pairing, union, and infinity, plus the following three schemata—in all three, the formula φ may have parameters.

1. Induction
   $[((\forall y \in x)\varphi(y) \to \varphi(x)) \to (\forall x)\varphi(x)]$,
   any φ

2. Δ_0-separation
   $(\forall v)(\exists y)(\forall x)[x \in y \leftrightarrow (x \in v \ \&\ \varphi(x)]$,
   φ has only bounded quantifiers

3. Δ_0-collection
   $(\forall u)[(\forall x \in u)(\exists y)\varphi(x, y) \to (\exists v)(\forall x \in u)(\exists y \in v)\varphi(x, y)]$,
   φ has only bounded quantifiers

T_{KP}

Fact. There is a computable tree $T_{KP} \subseteq \omega^{<\omega}$ whose paths represent the complete diagrams of ω-models of KP.

What do the ω-models of KP look like?

They include all hyperarithmetical sets. The computable ordinals form an initial segment of the ordinals. There may be further standard ordinals, and there may also be non-standard ordinals.

Comparing T_{PA} and T_{KP}

The ω-branching tree T_{KP} is in some ways similar to T_{PA}, and in other ways different.

One difference is that T_{KP} does not lie on top among ω-branching trees; in fact, no tree lies on top.

Binns-Simpson. For any computable tree $T_1 \subseteq \omega^{<\omega}$, with paths, there is a computable tree $T_2 \subseteq \omega^{<\omega}$, with paths, s.t. $T_2 \not\leq_w T_1$ (i.e., some path through T_1 does not compute a path through T_2).

Tree rank (foundation rank)

Definition. For a tree T and $\sigma \in T$,

(1) $rk(\sigma) = 0$ if σ has no successors,

(2) for $\alpha > 0$, $rk(\sigma) = \alpha$ if σ has successors, all of ordinal rank, and α is the first ordinal greater than the ranks of all successors of σ,

(3) $rk(\sigma) = \infty$ if σ does not have ordinal rank.

We define $rk(T)$ to be the rank of the top node $\emptyset$ in T.

Path or rank?

Fact (*ZFC*). For a tree $T \subseteq \omega^{<\omega}$, T has a path iff it is unranked.

Barwise. If T is a computable tree with no path, then $rk(T)$ is a computable ordinal.

Computable trees in ω-models of KP

An ω-model of KP calculates computable ordinal ranks just as we do. Suppose T is a computable tree. If $rk(T) = \alpha$ in the real world, then $rk(T) = \alpha$ in ω-models of KP. If T has no path in the real world, then there is no path in ω-models of KP.

Fact: The theorem saying that a tree has a path iff it is unranked may fail in ω-models of KP.

Proof.

Let T be a computable tree that has paths but no hyperarithmetical path. Then in $L_{\omega_1^{CK}}$, T is unranked, with no path. □

Trees with paths but no hyperarithmetical paths

(1) The tree T_{KP} has paths but no hyperarithmetical paths.

(2) For a Harrison ordering H, let T_H be the tree of decreasing sequences in H. Again T_H has paths but not hyperarithmetical paths.

Recall what is a Harrison ordering.

Harrison. There is a computable ordering of type $\omega_1^{CK}(1+\eta)$ with no hyperarithmetical decreasing sequence.

Such an ordering is a "Harrison ordering".

A special Harrison ordering

For later use, we consider a special Harrison ordering.

Goncharov-Harizanov-K-Shore.

(1) The Turing degrees of the well-ordered parts of Harrison orderings are the same as those of paths through O.

(2) There is a path through O that does not compute $\emptyset'$.

Consequence. For a Harrison ordering H in which the well-ordered part W does not compute $\emptyset'$, T_H has a path f that does not compute $\emptyset'$. Moreover, we may take f extending any finite decreasing sequence in $H - W$.

Existence of ω_1^{CK}

Theorem. In any ω-model of KP, the following are equivalent:

1. Every computable tree is ranked or has a path,
2. ω_1^{CK} exists; i.e., there is a first ordinal not isomorphic to any computable ordering.

Trees with non-standard rank

Proposition. There are ω-models of KP in which some computable trees have non-standard ordinal rank.

Proof.
Let T be a computable tree with paths but no hyperarithmetical path. Then T has nodes of all computable ordinal ranks. There is an ω-model M of KP s.t. some $\sigma \in T$ has non-standard rank. Then $T_\sigma = \{\tau : \sigma\tau \in T\}$ has non-standard rank. $\square$

Computing paths

Note. Let T be a computable tree, and let M ne an ω-model of KP in which T is unranked. Then $D^c(M)$ computes a path through T.

Theorem (Weisshaar). Let T_1, T_2 be computable trees, and let M be an ω-model of KP in which T_1, T_2 have non-standard rank, where $rk(T_1) \leq rk(T_2)$. If f is a path through T_1, then $f \oplus D^c(M)$ computes a path through T_2.

Can we drop $D^c(M)$?

Theorem. There are computable trees T_1, T_2 s.t. in some ω-model M of KP, $rk(T_1) < rk(T_2)$, and some path f through T_1 does not (by itself) compute a path through T_2.

Idea of proof.

Let T_H be the tree of decreasing sequences in a Harrison ordering H whose well-ordered part does not compute $\emptyset'$. Take an ω-model M of KP, in which some $\sigma \in T_H$ and $\tau \in T_{KP}$ both have non-standard rank, with $rk(\sigma) < rk(\tau)$. Let T_1 be the tree below σ in T_H, let T_2 be the tree below τ in T_H, and let f be a path through T_σ that does not compute $\emptyset'$. □

Is the ordering on ranks determined by the trees?

Theorem. There are computable trees T_1, T_2 and ω-models M_1, M_2 of KP s.t. $M_1 \models rk(T_1) < rk(T_2)$ and $M_2 \models rk(T_2) < rk(T_1)$.

Independence

Gödel-Rosser. For any computable set A of axioms (extending PA), let φ_A be the sentence that refers to itself, saying "for any proof of me from A, there is a smaller proof of my negation". If A is consistent, then so are $A \pm \varphi$.

We can use this to get the following.

Fact. There are $2^{\aleph_0}$ completions of PA.

Gödel-Rosser analogue. For a computable set A of axioms (extending KP), T_A is a computable tree whose paths represent the complete diagrams of ω-models of A. Let φ_A be the sentence that refers to itself, saying $rk(T_{A+\varphi}) \leq rk(T_{A+\neg\varphi})$. If A is ω-consistent, then so are $A \pm \varphi$.

Corollary. There are $2^{\aleph_0}$ ω-consistent completions of KP.

Number of models

Each completion of PA has $2^{\aleph_0}$ pairwise non-isomorphic models.

For ω-consistent completions of KP, the number of non-isomorphic ω-models varies.

(1) $Th(L_{\omega_1})$ has $2^{\aleph_0}$ non-isomorphic ω-models.

(2) $Th(L_{\omega_1^{CK}})$ has just one ω-model, up to isomorphism.

Non-minimality

Fact. For any completion T_1 of PA, there is another completion T_2 of strictly lower degree.

The analogous statement about complete diagrams of ω-models of KP is also true.

Theorem. Let M be an ω-model of KP. There is an ω-model N s.t. $D^c(N) <_T D^c(M)$.