Computing on Streams; Analog and Digital Models

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Analog and digital computation

Both process ***infinite data***

— typically ***real numbers***,

— originating as ***physical measurements***.

Digital computation:

— data represented by ***streams*** of ***discrete approximations***

— computations from ***input approx's*** to ***output approx's***

— computation is ***"exact"***

Analog computation:

— data rep'd by ***physical quantities*** (voltage, displacement, . . .)

— processed by ***networks*** of ***mechanical/electrical components*** in ***continuous time***

— computation is ***approximate***

Classical digital computation theory:

- Comprehensive, deep mathematical theory of digital computation (1930s: Turing, Gödel, Kleene, Church, $\ldots$)

- ***Generalized*** to computation on other structures, e.g. $\mathbb{R}$, $\mathcal{C}[\mathbb{T}, \mathbb{R}]$.

We will use "**tracking computability**" as paradigm of **digital comp.**

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\quad \boldsymbol{f} \quad} & \mathbb{R} \\ \boldsymbol{\alpha} \uparrow & & \uparrow \boldsymbol{\alpha} \\ \mathbb{N} & \xrightarrow{\quad \boldsymbol{\varphi} \quad} & \mathbb{N} \end{array}$$

Under "reasonable conditions" it is **equivalent** to:

— **Grzegorczyk-Lacomb** computability,

— **effective polynomial approximability**,

and likely:

— **Weihrauch's TTE**.

Analog computation theory:

- **Less developed**

  — Kelvin, Bush, Shannon . . .

- **Resurgence of interest**

  — Marion Pour-El, Olivier Bournez, Felix Costa, Daniel Graça, . . .

Why the resurgence of interest in analog computation?

- Interesting ***theoretical questions*** in

— ***computation theory*** + ***real analysis***

— interesting issues in ***philosophy of science***:
e.g., nature of physical measurements.

- But what ***practical use*** is it?

One answer:

“There is a perceived competition between “analog” and “digital”, but this . . . is a complete fallacy. Digital circuits rule the world. No one can deny the computational power of desktop computers, laptops, cell phones . . . However, a ***completely digital computer*** would be ***completely useless*** . . .

“To make a computer useful, we need video and audio inputs and outputs, which are ***analog*** . . .

“***Analog circuits*** allow you to listen to music and make your iPod more than a pretty paperweight . . .

“***You can build an entirely analog computer . . . but you can’t build an entirely digital computer.***”

— *Kent H. Lundberg, Introduction to Special Issue on the History of Analog Computing, IEEE Control Systems Magazine, June 2005*

General problem

To show that (or under what conditions) analog systems have ***solutions***, which are

- ***well-defined***, i.e., ***unique***,
- ***computable***, in the sense of ***classical (digital) computability theory***.
- ***stable***, i.e., ***continuous*** in the parameters.

Significance of continuity

Hadamard's principle (in the formulation of Courant and Hilbert):

> ***For a scientific problem to be well posed, the solution must (apart from existing and being unique) depend continuously on the data.***

Note:

Scientific measurement in the presence of noise is only possible under assumption of ***continuity of data***, to ensure ***repeatability*** and ***reliability*** of results.

Analog Network:

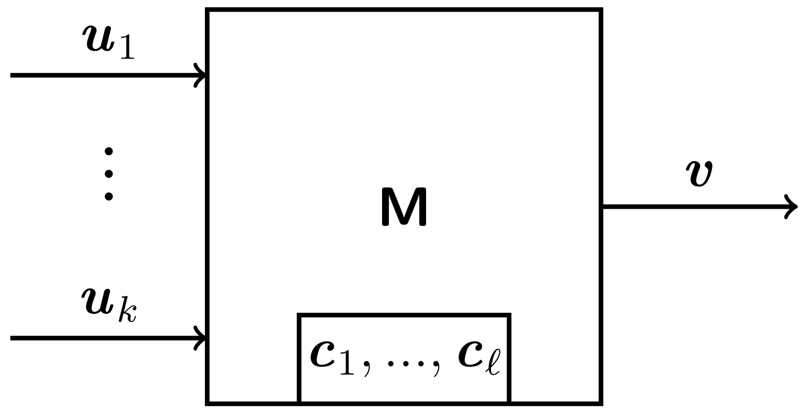
An arrangement of ***modules*** and ***channels*** carrying data from a ***complete*** (***separable***) ***metric space*** A.

- Operates in ***continuous time*** $\mathbb{T}$ ($=$ non-negative reals)

- ***Channels*** carry signals: ***continuous streams*** from A, i.e., continuous functions

$$u : \mathbb{T} \to A$$

- We work with the space $\mathcal{C}[\mathbb{T}, A]$ of ***continuous streams*** from A.



A module **M** has:

- locations for ***parameters*** $c_1, \ldots, c_l$.
- finitely many ***input channels*** $u_1, \ldots, u_k$,
- one ***output channel*** v,

Module function $\mathsf{F_M}\colon A^l \times \mathcal{C}[\mathbb{T}, A]^k \;\to\; \mathcal{C}[\mathbb{T}, A]$,

$$\mathsf{F_M}(\bar{c}, \bar{u}) \;=\; v$$

The m modules $\mathbf{M}_1, \ldots, \mathbf{M}_m$ with module functions $\mathbf{F}_1, \ldots, \mathbf{F}_m$, form a ***network*** **N** with

- ***parameters*** $\bar{c} = (c_1, \ldots, c_r) \in A^r$.
- ***input*** streams $\bar{x} = (x_1, \ldots, x_p) \in \mathcal{C}[\mathbb{T}, A]^p$,
- "***mixed***" streams $\bar{u} = (u_1, \ldots, u_m) \in \mathcal{C}[\mathbb{T}, A]^m$.

So **N** has a ***stream transformation function***

$$\mathsf{F}^{\mathsf{N}}: A^r \times \mathcal{C}[\mathbb{T}, A]^p \times \mathcal{C}[\mathbb{T}, A]^m \rightarrow \mathcal{C}[\mathbb{T}, A]^m$$

as a ***vector*** of the module functions $\mathbf{F}_1, \ldots, \mathbf{F}_m$:

$$\mathsf{F}^{\mathsf{N}}(\bar{c}, \bar{x}, \bar{u}) = (\mathsf{F}_1(\bar{c}'_1, \bar{x}'_1, \bar{u}'_1), \ldots, \mathsf{F}_m(\bar{c}'_m, \bar{x}'_m, \bar{u}'_m))$$

where $(\bar{c}'_i, \bar{x}'_i, \bar{u}'_i)$ are the ***sublists*** of $(\bar{c}, \bar{x}, \bar{u})$ ***local to*** $\mathbf{M}_i$.

So **N** has an ***equational specification***

$$v_i(t) = \mathsf{F}_i(\bar{c}'_i, \bar{x}'_i, \bar{u}'_i)(t) \qquad (i = 1, \ldots, m, \ t \geq 0) \qquad \textbf{(E)}$$

So **N** has an ***equational specification***

$$v_i(t) \;=\; \mathsf{F}_i(\bar{c}_i', \bar{x}_i', \bar{u}_i')(t) \qquad (i = 1, \ldots, m, \;\; t \geq 0) \qquad \textbf{(E)}$$

Then a solution of (**E**) is a ***fixed point*** of

$$\mathsf{F}^{\mathbf{N}}_{\bar{c},\bar{x}} \;=\; \mathsf{F}^{\mathbf{N}}(\bar{c}, \bar{x}, \,\cdot\,) \colon\, \mathcal{C}[\mathbb{T}, A]^m \;\to\; \mathcal{C}[\mathbb{T}, A]^m,$$

representing an ***equilibrium state*** for N.

We are esp. interested in ***stream operators*** like $\mathsf{F}^{\mathbf{N}}_{\bar{c},\bar{x}}$ that are ***contracting*** according to the metric on $\mathcal{C}[\mathbb{T}, A]$ —

Since then, by ***Banach's fixed-point theorem:***

Theorem 1 (Solution of network equations (E))

Suppose $\mathsf{F}^{\mathsf{N}}_{\bar{c},\bar{x}}$: $\mathcal{C}[\mathbb{T}, A]^m \;\rightarrow\; \mathcal{C}[\mathbb{T}, A]^m$

is ***contracting*** at $(\bar{c}, \bar{x}) \in A^r \times \mathcal{C}[\mathbb{T}, A]^p$.

Then there is a unique stream tuple

$$\bar{u} \;=\; \mathsf{FP}(\mathsf{F}^{\mathsf{N}}_{\bar{c},\bar{x}})$$

satisfying (**E**).

- Now consider this **fixed point** $\bar{u}$ as a ***function of*** $\bar{c}$,$\bar{x}$.

Recall **Hadamard's Principle**.

Continuity and Computability of FP operation

(John Tucker, Nick James, JZ)

Theorem 2 (Continuity of FP operation)
Suppose $\boldsymbol{F}_{\bar{\boldsymbol{c}},\bar{\boldsymbol{x}}}$ is ***contracting*** and ***continuous*** in $(\bar{\boldsymbol{c}},\bar{\boldsymbol{x}})$.
Then FP($\boldsymbol{F}_{\bar{\boldsymbol{c}},\bar{\boldsymbol{x}}}$) is ***continuous*** in $(\bar{\boldsymbol{c}},\bar{\boldsymbol{x}})$.

Theorem 3 (Tracking computability of FP).

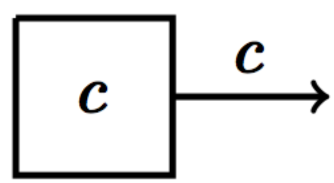
Suppose $\boldsymbol{F}_{\bar{\boldsymbol{c}},\bar{\boldsymbol{x}}}$ satisfies conditions of Thm 2, and ***further***:
$\boldsymbol{F}_{\bar{\boldsymbol{c}},\bar{\boldsymbol{x}}}$ is ***tracking computable***.
Then **FP**($\boldsymbol{F}_{\bar{\boldsymbol{c}},\bar{\boldsymbol{x}}}$) is ***tracking computable*** in $(\bar{\boldsymbol{c}},\bar{\boldsymbol{x}})$.

The Shannon GPAC

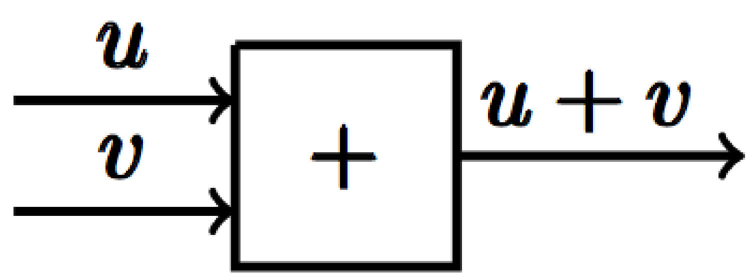
We consider the ***General Purpose Analog Computer*** (Shannon 1941).

It has 4 basic modules:

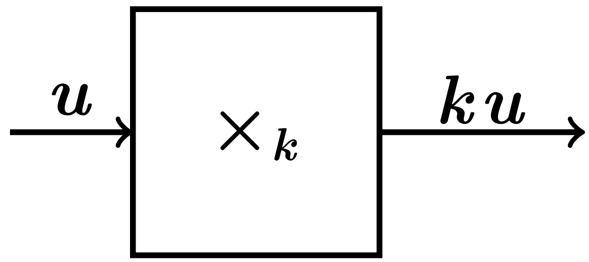
- **constant:**



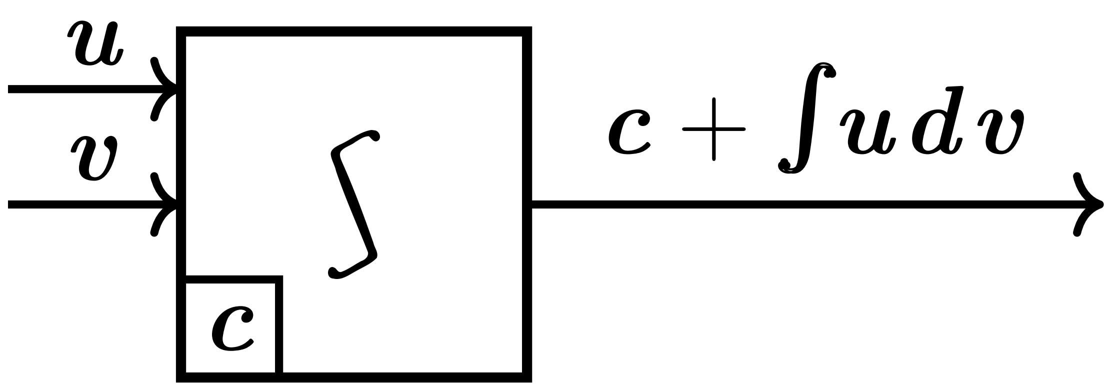
- **adder:**



- **scalar multiplier:**



- **(Stieltjes) integrator:**



Semantics of the GPAC

In general, a GPAC $\mathcal{G}$ defines a ***"fixed-point function"***

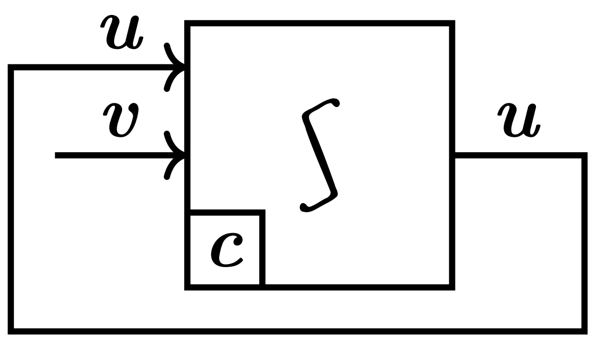
$$F\colon \mathbb{R}^r \times \mathcal{C}[\mathbb{T},\mathbb{R}]^p \;\to\; \mathcal{C}[\mathbb{T},\mathbb{R}]^q$$

the **fixed point(s)** of which, i.e., the value(s) of the mixed channel(s), give the computed function G.

$\mathcal{G}$ is ***well-posed*** on an open $U \subseteq$ ***dom***$(\mathcal{G})$ if F

- **exists** on U, and is
- **unique** and
- **continuous** on U.

A simple example:



Here $(r=1,\ p=1,\ q=1)$ we have

$$F(c,v,u) \;=\; c+\int_0^t u(s)dv(s) \;=\; u(t).$$

Differentiating both sides:

$$u'(t) \;=\; u(t)\,v'(t).$$

This is a linear ODE with solution

$$u(t) \;=\; c\,\mathrm{exp}\big(v(t)-v(0)\big).$$

So this GPAC is **well-posed** on its domain.

Characterizing GPAC-computability

A function $f\colon \mathbb{T} \to \mathbb{R}$ is **differentially algebraic** on $U \subseteq \mathbb{T}$
if $f \in C^k(\mathbb{T})$ for some k, and satisfies

$$P(t, f(t), f'(t), \ldots, f^{(k)}(t)) = 0$$

for some polynomial P in $k+2$ variables, and all $t \in U$.

Theorem 4 ***(Shannon, Pour-El, Lipshitz & Rubel, Graça & Costa)***.
Let $\mathcal{G}$ be a Shannon GPAC well-posed on some open $U \subseteq$ ***dom***$(\mathcal{G})$.
Then the function computed by $\mathcal{G}$ is **differentially algebraic** on U.

A problem with GPACs:

The **gamma function**

$$\Gamma(t) \;=\; \int_0^\infty x^{t-1} e^{-x} dx$$

is ***not diff. alg.*** , and so ***cannot be GPAC-computable***.

This is a symptom of a **more general problem**:
the Shannon GPAC can reason about real-valued functions of **only one independent variable** (“time" t).

Replacing the **input space** $\mathcal{C}[\mathbb{T}, \mathbb{R}]$

by $\mathcal{C}[\mathbb{T}, \mathbb{R}] \;\times\; \ldots \;\times\; \mathcal{C}[\mathbb{T}, \mathbb{R}]$

does not work — even at a conceptual level!

The solution (***Diogo Poças***, thesis) is to **replace** the **output space**

$$\mathcal{C}[\mathbb{T}, \mathbb{R}] \quad \text{by} \quad \mathcal{C}[\mathbb{T}, X]$$

where X may be a **function space**
(e.g. $X = \mathcal{C}[D, \mathbb{R}]$ for a suitable domain $D \subset \mathbb{R}^n$).

Then the channels carry X-**valued streams of data**

$$u : \mathbb{T} \to X$$

which under the **uncurrying**

$$u : \mathbb{T} \to (D \to \mathbb{R}) \simeq \mathbb{T} \times D \to \mathbb{R}.$$

correspond to functions of $n+1$ real variables: $t \in \mathbb{T}$, $x \in D$.

This suggests a **generalization** of the GPAC to handle **many-sorted** data.

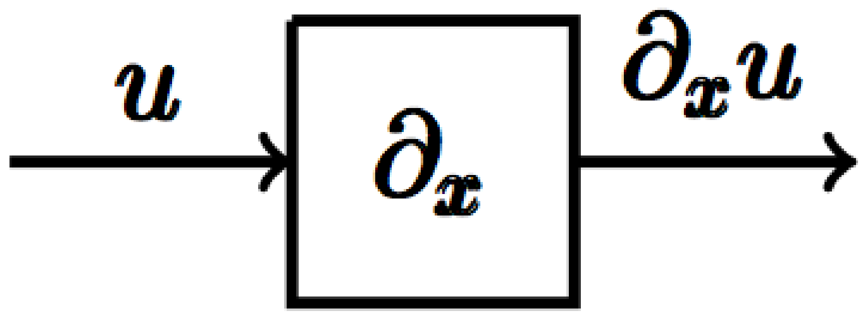
An X-**GPAC** has **channels** containing functions of $\boldsymbol{x} \in \boldsymbol{X}$ as well as $\boldsymbol{t}$.

It has the 4 **basic modules** of the GPAC:

- **constant** — w.r.t. **time** $\boldsymbol{t}$, i.e. a function of $\boldsymbol{x}$ (only)!
- **adder**
- **scalar multiplier**
- **Stieltjes integrator** — integrates w.r.t. $\boldsymbol{t}$.

In addition, there is a

- **differential module** $\boldsymbol{\partial_x}\colon \boldsymbol{X} \rightharpoonup \boldsymbol{X}$



Depending on X ($= \mathcal{C}(\mathbb{R}), \mathcal{C}^1(\mathbb{R}), \mathcal{C}^\infty(\mathbb{R}), \ldots$),

the operator ∂_x is (in general) **partial**, and (**not continuous**, but) **closed**, i.e.

If $f_n \in$ ***dom***(∂_x) and $f_n \to f$ and $\partial_x(f_n) \to g$

then $f \in$ ***dom***(∂_x) and $\partial_x(f) = g$.

Hence for X-**GPACs**, we change our definition (p. 18) of **"well-posedness"** of $\mathcal{G}$ on $U \subseteq$ ***dom***$(\mathcal{G})$ by

- replacing **"continuous"** by **"closed on U"**.

Note also:

- The function space X can now be represented (in general) not as a Banach space, but as a Fréchet space.

Characterizing X-GPAC computability

A **partial differential algebraic system (PDAS)**
on n variables $u_1, \ldots . u_n$ (representing network channels)
is a finite set of polynomial identities of the form (cf. p. 20)

$$P(t, u_1, \ldots, u_n, \partial_x^{\alpha_1} u_1^{(\beta_1)}, \ldots, \partial_x^{\alpha_n} u_n^{(\beta_n)}) = 0,$$

together with **initializing equations** for these variables:

$$u_k^{(\beta)}(0) = 0.$$

As with X-**GPACs** (p. 25), we define:

A **PDAS** is ***well-posed*** on U if it has a **solution** on U that is

- **unique** and
- defines a **closed operator**.

Theorem 5 ***(Poças)***.
Let $\mathcal{G}$ be an X-**GPAC**, well-posed on some $U \subseteq$ ***dom***$(\mathcal{G})$.
Then the function computed by $\mathcal{G}$ is the solution of a **PDAS** well-posed on U. (And conversely.)

- ***Consequences of Thm 5*** ...

- *The good news*:
  From the proof of Thm 5 (converse direction), we can construct X-**GPACs** that solve PDEs for functions $u(x,t)$ where the variables 'x' and 't' can be **separated**, e.g

(1-dim) heat equation: $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

wave equation: $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

transport equation: $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$$

- *The bad news*:
  From Thm 5, we also see that the gamma function is
  ***not* X-GPAC *computable*!**

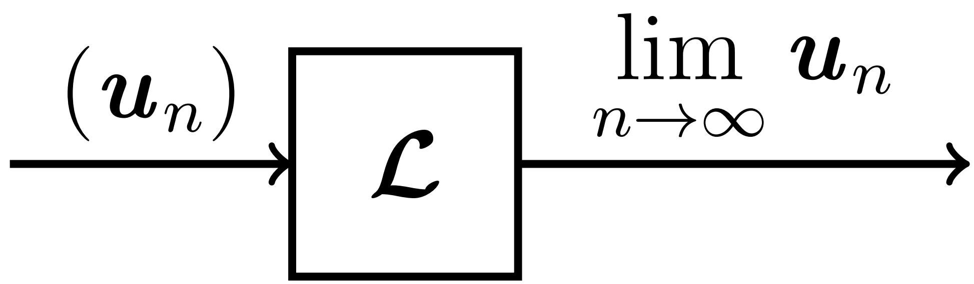
Q. What is missing from the X-**GPAC**?

A. The concept of ***limits*** of sequences of functions.

So we introduce the ***“limit"*** X-GPAC: $\mathcal{L}X$-**GPAC**, with the (partial) **limit operation**

$$\boldsymbol{\mathcal{L}} : \boldsymbol{X}^{\mathbb{N}} \rightharpoonup \boldsymbol{X}$$

and the module



This is a ***partial*** operation —

defined only on (effective) ***Cauchy sequences*** of functions $(\boldsymbol{u}_n)$.

Computability of the Gamma Function

The gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

is known to be **not diff. alg.**, and hence **not GPAC-computable**.

However it can be written as the limit of a sequence of integrals:

$$\Gamma(x) = \lim_{n \to \infty} \int_{1/n}^{n} \ldots \, dt$$

where the integrals on the r.h.s. are X-**GPAC** computable.

Hence $\Gamma(x)$ is $\mathcal{L}X$-**GPAC** computable.

Similarly, the ***Riemann zeta function***

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

which (for real $x \geq 2$) can be re-written as:

$$\zeta(x) = \frac{2^x}{x-1} - 2^x \int_0^{\infty} \frac{\sin(x \arctan t)}{(1+t^2)^{x/2}(e^{\pi t+1})} dt$$

can be shown (by a similar technique) to be $\mathcal{L}X$-**GPAC** computable.

AN IMPORTANT PROBLEM:
Comparing the strengths of Analog and Digital Models

We know:
Analog comp. is (in general) ***weaker than*** **digital** comp.:

- **GPAC** computability $\Longrightarrow$ **tracking** computability,
  $\not\Longleftarrow$

- X-**GPAC** computability $\Longrightarrow$ **tracking** computability.
  $\not\Longleftarrow$

- What about $\mathcal{L}X$-**GPAC** ???

Poças showed:

- $\mathcal{L}X$-**GPAC** computability $\Longrightarrow$ **tracking** computability,

but could not prove the converse.

Conjecture:

$\mathcal{L}X$-**GPAC** computability $\underset{\not\Longleftarrow}{\Longrightarrow}$ **tracking** computability

(under “reasonable" assumptions).

Assuming this conjecture is true ...

Open Problem:

To find an ***augmented version*** $\mathcal{L}X$-**GPAC**$^+$ of $\mathcal{L}X$-**GPAC** s.t.

$\mathcal{L}X$-**GPAC**$^+$ computability $\iff$ **tracking** computability

References

On my web site, look at my publications in "Area 4: Computability":

http://www.cas.mcmaster.ca/ zucker/Pubs/areas.html#4-comp

— esp. papers 17, 20, 21, 22, 27, 29.

Thank you!